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CONFIGURATION SPACE REPRESENTATION FOR NON-LINEAR FIELDS

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Introduction.

In the theory of nucleons and mesons we deal with a situation in which the coupling between the two fields is not small. It is, therefore, of importance for the treatment of such problems to develop methods more powerful than perturbation theory. The divergence difficulties inherent in current field theory necessitate a formulation of the non-perturbation approaches which allow for an incorporation of the idea of renormalization of mass and charge. In practice, this implies as a necessary condition that the formalism must be covariant.

The method proposed by Salpeter and Bethe [1], [10] for the treatment of the two-body problem is an example of such an approach. A general theory of a similar kind has been initiated by Schwinger [2]. In this theory, one starts from the consideration of certain combinations of vacuum expectation values of time ordered products of field operators, the so-called Green's functions. In general such quantities obey inhomogeneous equations of motion. It can be seen that the study of the oscillating solutions of the corresponding homogeneous equations provides information about the energy and momentum values of stationary states of the system. According to Schwinger, these homogeneous equations apply to scattering problems as well. By his method, equations of the Bethe-Salpeter type can be established without reference to the limit of no interaction. However, it seems rather difficult by means of this kind of approach to obtain a clear understanding of the nature of the wave functions which obey the homogeneous equation.

Partly to overcome this problem, Heisenberg [5] and Freese [4] have proposed to start directly from a definition of the wave function for the problem. In the general formalism developed by Freese it is shown how, for each state of the system, one can construct an infinite set of wave functions from free field Green's functions and matrix elements of time ordered products of field operators. The construction is such that the discontinuities in the matrix elements are compensated by corresponding discontinuities in the free field Green's functions. Consequently, Freese's wave functions obey homogeneous equations of motion. The infinite set of wave functions constitutes a generalization of the Fock representation in the configuration space for free fields to the case of interacting fields. For some problems one can substitute the infinite set of wave functions by essentially one function, only. The equation obtained for this function is of a similar structure as the equation of the Bethe-Salpeter type following from Schwinger's theory, but is in general not identical with Schwinger's equation. One reason for this may be found in the fact that free field concepts enter in Freese's representation.

In the present paper, an attempt is made at modifying the ideas of Heisenberg and Freese so as to unify their theory with that of Schwinger, and thus to combine the advantages of both formalisms. To this purpose, we employ the technique of variation of external sources developed by Peierls and Schwinger [6]. In Section 1, a survey of this method is given in a form which is convenient for our purpose. In Section 2, we relate to any state of the system a functional of the sources. The variational derivatives of this functional with respect to the sources define an infinite set of amplitudes. These are shown in Section 3 to generalize the Fock representation to non-linear fields. No reference to free field concepts is made in the definition of the state vector amplitudes. Several simple properties of the Fock representation are maintained in the non-linear case.

The problem of the construction of the scalar product of two states given in this configuration space representation has not been solved. Until further progress is made one must, therefore, use the term representation with some reservation. The equations of motion, in the configuration space representation, are derived in Section 4. Finally, in Section 5, a preliminary discussion is given of the one-nucleon problem and of the two-nucleon problem.

The corresponding equations of motion become identical with those following from Schwinger's theory.

Much of the discussion given by Freese can directly be taken over to the present formalism and is not repeated here. In particular for the discussion of scattering problems, the reader may be referred to Freese's paper.

All considerations below are of a highly formal character in so far as we have completely neglected the divergence difficulties. However, the renormalization theory, for instance in the form given by Källén [8], can easily be incorporated in the present formalism.

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1. The field equations including coupling to external sources.

With the aim to illustrate the general method we consider the example of a spin one-half field (nucleons) coupled to a scalar neutral meson field. With a suitable symmetrization of the interaction terms, the equations of motion are

$$\left.\begin{aligned} (\partial + M)\,\psi_0(x) + (\lambda/2)\left\{ u_0(x), \psi_0(x) \right\} &= 0, \\ (\bar{\partial} + M)\,\bar{\psi}_0(x) + (\lambda/2)\left\{ u_0(x), \bar{\psi}_0(x) \right\} &= 0, \\ (-\Box + m^2)\,u_0(x) + (\lambda/2)\left[\bar{\psi}_0(x), \psi_0(x)\right] &= 0. \end{aligned}\right\} \quad (1.1)$$

Here, λ is the coupling parameter, and ∂ and $\bar{\partial}$ denote

$$\partial = \gamma_\mu\,\partial/\partial x_\mu, \quad \bar{\partial} = -\,\gamma_\mu^T\,\partial/\partial x_\mu,$$

where γ_μ^T is the transposed of the matrix γ_μ. The index μ runs from one to four and $x_\mu = (x_1, x_2, x_3, x_4)$, $x_4 = it$. As usual, $\overline{\psi}_0$ is defined in terms of ψ_0^*, the Hermitian conjugate of ψ_0, as $\overline{\psi}_0 = \psi_0^* \gamma_4$. The units chosen are such that $\hbar = c = 1$.

As mentioned in the introduction, we employ the method of variation of external sources developed by Peierls and Schwinger [6]. Therefore, we introduce external sources for all three kinds of fields and thus modify the equations (1.1) to

$$\left.\begin{aligned} (\partial + M)\,\psi(x) + (\lambda/2)\left\{u(x), \psi(x)\right\} + \varphi(x) &= 0,\\ (\overline{\partial} + M)\,\overline{\psi}(x) + (\lambda/2)\left\{u(x), \overline{\psi}(x)\right\} + \overline{\varphi}(x) &= 0,\\ (-\Box + m^2)\,u(x) + (\lambda/2)\left[\overline{\psi}(x), \psi(x)\right] + I(x) &= 0. \end{aligned}\right\} \quad (1.2)$$

By omitting in these equations the subscript attached to the field operators in (1.1) we distinguish the source-dependent field variables from the usual ones describing the closed system. In the following, we assume that the sources vanish for both $|\vec{x}|$ and $|t|$ tending to infinity. With this restriction the equations (1.2) can be supplemented by a boundary condition which requires that the source-dependent field operators become identical with the usual source-free fields in the infinite past. Considering such solutions only we can regard the field variables as functionals of the sources. As no other solutions of the equations (1.2) will be considered in the following, it is superfluous to discriminate by any label this retarded solution from other possible ones.

We take $I(x)$, the external source of the meson field, as a c-number. Of course, one could also treat the external spinor sources as c-numbers. However, this is not what we shall do. In order that the external sources be useful, one should take the spinor sources as the analogue of c-numbers for the fermion case, i. e. as quantities such that

$$\left\{\varphi(x), \varphi(x')\right\} = \left\{\varphi(x), \overline{\varphi}(x')\right\} = \left\{\overline{\varphi}(x), \overline{\varphi}(x')\right\} = 0, \quad (1.3)$$

and

$$\left.\begin{aligned} \left\{\varphi(x), \psi_0(x')\right\} = \left\{\overline{\varphi}(x), \psi_0(x')\right\} &= 0\\ \left\{\varphi(x), \overline{\psi}_0(x')\right\} = \left\{\overline{\varphi}(x), \overline{\psi}_0(x')\right\} &= 0, \end{aligned}\right\} \quad (1.4)$$

while φ and $\overline{\varphi}$ commute with I (and of course with any other c-number). For the further specification of the manifold of pairs of spinor sources it is advantageous to write φ and $\overline{\varphi}$ in the form

$$\left.\begin{aligned}\varphi(x) &= \Theta_0 f(x),\\ \overline{\varphi}(x) &= \Theta_0 g(x),\end{aligned}\right\} \tag{1.5}$$

where Θ_0 is a constant operator which commutes with f and g and anticommutes with ψ_0 and $\overline{\psi}_0$. Hence, due to (1.4) and (1.3), f and g commute with ψ_0 and $\overline{\psi}_0$ and satisfy

$$\{f(x), f(x')\} = \{f(x), g(x')\} = \{g(x), g(x')\} = 0. \tag{1.6}$$

As is well known, essentially only one such quantity Θ_0 exists, viz. the parity of the difference ΔN between the number of nucleons and the number of anti-nucleons. In terms of the field operators, ΔN is

$$\Delta N = \frac{1}{2}\int\left[\psi_0^*(\vec{x}, t), \psi_0(\vec{x}, t)\right] d^3\vec{x}. \tag{1.7}$$

We choose Θ_0 as

$$\Theta_0 = (-1)^{\Delta N} = \text{parity of } \Delta N, \tag{1.8}$$

thereby normalizing Θ_0 so that $\Theta_0^2 = 1$ and $\Theta_0 \mid 0\rangle = \mid 0\rangle$, where $\mid 0\rangle$ is the vacuum state of the source-free system.

Corresponding to any pair f, g, we define the domain of pairs of sources obtained by allowed variations as the totality of pairs of the form $f + \delta f$, $g + \delta g$, where δf and δg are infinitesimal and anti-commute with f and g, i. e.

$$\left.\begin{aligned}\{\delta f(x), f(x')\} &= \{\delta g(x), f(x')\} = 0,\\ \{\delta f(x), g(x')\} &= \{\delta g(x), g(x')\} = 0.\end{aligned}\right\} \tag{1.9}$$

It should be noted that (1.9) is not a consequence of (1.6). For any pair f, g we now require the manifold of allowed variations to be so large that we, from a relation of the type

$$\int\left[\delta f(x) K(x) + \delta g(x) L(x)\right] d^4x = 0, \tag{1.10}$$

holding for all pairs of allowed variations, can conclude that $K(x)$ and $L(x)$ vanish identically. In (1.10), K and L are considered as quantities of the same nature as f and g.

This last mentioned property of the spinor sources, together with (1.6), is all we need for the formal calculations below. The consistency of all requirements is demonstrated in Appendix I by the construction of an example of a possible domain of pairs f, g. As shown there, one can imagine the quantities f, g, or as we shall say, the f-number pairs, to be infinite matrices. It should, however, be emphasized that the f-number pairs will be treated as a kind of numbers and not as operators. In other words, all matrix elements are matrix elements in the space of the source-free operators only, and are, for the rest, quantities of the same nature as the f-number pairs. Thus, corresponding to (1.5) and the fact that the parity of the vacuum state of the source-free system is unity, we write

$$\langle 0 \mid \varphi(x) \mid \Psi \rangle = f(x) \langle 0 \mid \Theta_0 \mid \Psi \rangle = f(x) \langle 0 \mid \Psi \rangle . \qquad (1.11)$$

In this relation $\mid \Psi \rangle$ can be any state.

By means of the field equations one can easily see that the spinor sources anticommute with the source-dependent spinor fields and commute with u. This statement is based on the essential property of the source-dependent fields that ψ and $\overline{\psi}$ are odd functionals of quantities which anticommute with the spinor sources, while u is an even functional of such quantities. Thus

$$\left.\begin{aligned} \{\varphi(x), \psi(x')\} = \{\overline{\varphi}(x), \psi(x')\} = 0, \\ \{\varphi(x), \overline{\psi}(x')\} = \{\overline{\varphi}(x), \overline{\psi}(x')\} = 0. \end{aligned}\right\} \qquad (1.12)$$

Similarly, it can be verified that allowed variations $\delta\varphi$ and $\delta\overline{\varphi}$ anticommute with ψ and $\overline{\psi}$ and commute with u. By allowed variations we here understand variations of the form $\delta\varphi = \Theta_0 \delta f$ and $\delta\overline{\varphi} = \Theta_0 \delta g$, where δf and δg satisfy (1.9).

Conversely, we could also have started from (1.12) instead of (1.4), as (1.4) follows from (1.12), the field equations (1.2) and the retarded boundary condition.

One further remark may be useful here. It can easily be verified from (1.2) and the boundary condition that the canonical commutation relations

$$\left.\begin{aligned}\{\psi_\alpha(\vec{x},t),\overline{\psi}_\beta(\vec{x}',t)\} &= (\gamma_4)_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')\\ [u(\vec{x},t),\dot{u}(\vec{x}',t)] &= i\,\delta(\vec{x}-\vec{x}')\end{aligned}\right\}\tag{1.13}$$

hold in the source-dependent case also.

We can now formulate the following main theorem as regards the dependence of the fields on the sources. For any infinitesimal variation δI of the meson field source, and for any pair of allowed variations $\delta\varphi$ and $\delta\overline{\varphi}$ of the spinor sources, the corresponding variations of the fields are given by

$$\left.\begin{aligned}\delta\psi(x) &= i\,[\int_{-\infty}^{t}\delta W(x')\,d^4x',\psi(x)],\\ \delta\overline{\psi}(x) &= i\,[\int_{-\infty}^{t}\delta W(x')\,d^4x',\overline{\psi}(x)],\\ \delta u(x) &= i\,[\int_{-\infty}^{t}\delta W(x')\,d^4x',u(x)],\end{aligned}\right\}\tag{1.14}$$

where the infinitesimal operator δW is

$$\delta W(x) = \delta\overline{\varphi}(x)\,\psi(x) + \overline{\psi}(x)\,\delta\varphi(x) + u(x)\,\delta I(x). \tag{1.15}$$

The statement (1.14) is included in the general variation principle for quantized systems formulated by Schwinger [6]. It is, however, quite easy to prove (1.14) directly from the field equations. Evidently, (1.14) is in accordance with the boundary condition. Therefore, we only need to show that the variations (1.14) satisfy the varied field equations. For instance, from the first equation (1.14), we get

$$\begin{aligned}(\partial + M)\,\delta\psi(x) &= i[\int_{-\infty}^{t}\delta W(x')\,d^4x',(\partial+M)\,\psi(x)]\\ &\quad + [\int d^3\vec{x}'\,\delta W(\vec{x}',t),\gamma_4\,\psi(x)].\end{aligned}$$

which by the field equations, the properties of the sources, and the canonical commutators becomes

$$\begin{aligned}(\partial + M)\,\delta\psi(x) + (\lambda/2)\{u(x),\,\delta\psi(x)\}\\ + (\lambda/2)\{\delta u(x),\,\psi(x)\} + \delta\varphi(x) = 0.\end{aligned}$$

This is precisely the equation one would have obtained by varying the first equation (1.2). In a similar manner one obtains the other varied field equations, and this verifies (1.14).

In concluding this section we shall reexpress the contents of the variational equations (1.14), using the notion of variational derivatives. Consider a functional, $\Phi\,[\varphi,\bar{\varphi},I]$ say, of the sources. Assume, that one can write the variation of this functional in the form

$$\delta\Phi\,[\varphi,\bar{\varphi},I] = \int \bigl(\delta f(x)\,A(x) + \delta g(x)\,B(x) + \delta I(x)\,C(x)\bigr)\,d^4x = 0,$$

holding for any infinitesimal allowed variations of the sources. Then, the quantities A, B, and C are uniquely determined. This follows from the conclusion drawn from (1.10). We can thus define A, B, and C as the variational derivatives of the functional Φ corresponding to variations of f, g, and I, respectively. It is convenient to introduce the notation

$$\begin{aligned}A(x) &= \delta\Phi\,[\varphi,\bar{\varphi},I]/\delta f(x),\\ B(x) &= \delta\Phi\,[\varphi,\bar{\varphi},I]/\delta g(x),\\ C(x) &= \delta\Phi\,[\varphi,\bar{\varphi},I]/\delta I(x).\end{aligned}$$

It should be emphasized that, for instance, $\delta f(x)$ and $A(x)$ do not commute in general. The variational derivatives introduced here are thus left-hand derivatives. In a similar way, one could introduce right-hand variational derivatives.

As above, let $|\,0\,\rangle$ be the vacuum state of the source-free system and let $|\,\Psi\,\rangle$ be any other source-independent state. From (1.14) we get

$$\delta\langle 0\,|\,\psi(x)\,|\,\Psi\rangle = i\int_{-\infty}^{\infty}\frac{1-\varepsilon(x'-x)}{2}\,\cdot$$
$$\times\langle 0|\,[\bar{\varphi}(x')\,\delta\varphi(x') + \delta\bar{\varphi}(x')\cdot\psi(x') + \delta I(x')\cdot u(x'),\,\psi(x)]\,|\Psi\rangle\,d^4x'.$$

As usual, $\varepsilon(x' - x)$ is the step function $(t' - t)/|t' - t|$. From this equation we infer, using the properties of allowed variations and relations like

$$\langle 0 \mid \delta\varphi(x) = \delta f(x) \langle 0 \mid \Theta_0 = \delta f(x) \langle 0 \mid, \tag{1.16}$$

that

$$\left.\begin{aligned} i\frac{\delta\langle 0|\psi(x)|\Psi\rangle}{\delta g(x')} &= -\frac{1-\varepsilon(x'-x)}{2}\langle 0|\{\psi(x'),\psi(x)\}|\Psi\rangle, \\ -i\frac{\delta\langle 0|\psi(x)|\Psi\rangle}{\delta f(x')} &= -\frac{1-\varepsilon(x'-x)}{2}\langle 0|\{\overline{\psi}(x'),\psi(x)\}|\Psi\rangle, \\ i\frac{\delta\langle 0|\psi(x)|\Psi\rangle}{\delta I(x')} &= -\frac{1-\varepsilon(x'-x)}{2}\langle 0|[u(x'),\psi(x)]|\Psi\rangle. \end{aligned}\right\} \tag{1.17}$$

In a similar manner, one may obtain expressions for the variational derivatives of matrix elements of the other field variables. The minus sign on the left-hand side of the second equation originates from the reordering $\overline{\psi}\delta\varphi = -\delta\varphi\overline{\psi}$ necessary to obtain the left-hand variational derivative with respect to f.

2. Generating functionals for ordered products of field operators.

The ordered products considered in this section can all be constructed from one operator $\boldsymbol{T}$ which, as we shall see, is the generator of the time ordered product as defined by Wick [9].

i) *The time ordered product.*

We introduce an operator $\boldsymbol{T}$ by the variational equation

$$\delta\,\boldsymbol{T} = -i\,\boldsymbol{T}\int_{-\infty}^{\infty}\delta\,W(x)\,dx \tag{2.1}$$

and the boundary condition $\boldsymbol{T} = 1$ in the limit of vanishing sources*. The infinitesimal operator δW, defined by (1.15), is closely connected to the total variation of the operator

$$W(x) = \overline{\psi}(x)\,\varphi(x) + \overline{\varphi}(x)\,\psi(x) + I(x)\,u(x). \tag{2.2}$$

It follows from the properties of the sources that

$$[\delta W(x'), \varphi(x)] = [\delta W(x'), \overline{\varphi}(x)] = [\delta W(x'), I(x)] = 0, \tag{2.3}$$

whence, by (1.14),

$$\delta_{\text{total}} W(x) = \delta W(x) + i\int_{-\infty}^{x} [\delta W(x'), W(x)]\, dx'. \tag{2.4}$$

We shall verify in detail that the solution of the variational equation (2.1) is given by

$$\left.\begin{aligned} \boldsymbol{T} &= \overline{P}\exp\left\{-i\int_{-\infty}^{\infty} W(x)\,dx\right\} \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dx' \cdots \int_{-\infty}^{\infty} dx^{(n)}\,\overline{P}\left\{W(x')\cdots W(x^{(n)})\right\}, \end{aligned}\right\} \tag{2.5}$$

where $\overline{P}$ orders the W-factors in the reverse sense of Dyson's [3] chronologically ordering operator. Thus, if $x^{(\mu)}$ antedates $x^{(\nu)}$, then $W(x^{(\nu)})$ appears to the right of $W(x^{(\mu)})$ in the $\overline{P}$-ordered product. To prove (2.5) we first consider the variation of an ordinary product of W-factors. By (2.4) we get

$$\left.\begin{aligned} &\delta\left\{W(x')\,W(x'')\ldots W(x^{(n)})\right\} \\ &\quad = i\left(\int_{-\infty}^{x'}\delta W(x)\,dx\right) W(x')\,W(x'')\ldots W(x^{(n)}) \\ &\quad + i\,W(x')\left(\int_{x'}^{x''}\delta W(x)\,dx\right) W(x'')\ldots W(x^{(n)}) \\ &\quad + \ldots \\ &\quad + i\,W(x')\,W(x'')\ldots W(x^{(n)})\left(\int_{x^{(n)}}^{\infty}\delta W(x)\,dx\right) + \end{aligned}\right\} \tag{2.6}$$

* For a more general discussion it might be of advantage to consider another solution of the variational equation (2.1) corresponding to the boundary condition $\boldsymbol{T} = S$ for $\varphi = \overline{\varphi} = I = 0$, where S is the S-matrix for the closed system. All considerations in the following remain valid for this choice of solution.

$$\left.\begin{aligned}&+\delta W(x')\,.\,W(x'')\ldots W(x^{(n)})\\&+\ldots\\&+W(x')\,W(x'')\ldots\delta W(x^{(n)})\\&-i\,W(x')\,W(x'')\ldots W(x^{(n)})\cdot\int_{-\infty}^{\infty}\delta W(x)\,dx,\end{aligned}\right\}\quad(2.6)$$

where we have collected the contributions from the commutators between δW and W in an obvious manner. The complete symmetry of the $\overline{P}$-ordered product allows us to write the variation of the general term in series (2.5) in the somewhat simpler form

$$\left.\begin{aligned}&\delta\int_{-\infty}^{\infty}dx'\ldots\int_{-\infty}^{\infty}dx^{(n)}\,\overline{P}\left\{W(x')\,W(x'')\ldots W(x^{(n)})\right\}\\&=i\int_{-\infty}^{\infty}dx'\ldots\int_{-\infty}^{\infty}dx^{(n)}\int_{-\infty}^{\infty}dx\,\overline{P}\left\{\delta W(x)\,W(x')\ldots W(x^{(n)})\right\}\\&-i\int_{-\infty}^{\infty}dx'\ldots\int_{-\infty}^{\infty}dx^{(n)}\,\overline{P}\left\{W(x')\ldots W(x^{(n)})\right\}\cdot\int_{-\infty}^{\infty}\delta W(x)\,dx\\&+n\int_{-\infty}^{\infty}dx'\ldots\int_{-\infty}^{\infty}dx^{(n)}\,\overline{P}\left\{\delta W(x')\cdot W(x'')\ldots W(x^{(n)})\right\}.\end{aligned}\right\}\quad(2.7)$$

If we introduce this expression into the variation of $\boldsymbol{T}$ obtained from (2.5), we see that the contributions from the first and the third term on the right-hand side of (2.7) cancel, and that the sum of the remaining terms equals the right-hand side of (2.1). This verifies (2.5) as this expression obviously is in accordance with the boundary condition.

All allowed variations commute with W. Thus, by (2.5), also $\boldsymbol{T}$ commutes with these variations and we can write (2.1) in the form

$$\delta\boldsymbol{T}=-i\int_{-\infty}^{\infty}dx\left\{\delta\overline{\varphi}\,\boldsymbol{T}\psi-\delta\varphi\,\boldsymbol{T}\overline{\psi}+\delta I\,\boldsymbol{T}u\right\}.\qquad(2.8)$$

Consequently, for any source-independent state $|\,\Psi\rangle$, we have by (1.16)

$$\left.\begin{aligned} i\frac{\delta\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle}{\delta I(x)} &= \langle 0\,|\,\boldsymbol{T}u(x)\,|\,\Psi\rangle \\ i\frac{\delta\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle}{\delta g(x)} &= \langle 0\,|\,\boldsymbol{T}\,\psi(x)\,|\,\Psi\rangle \\ -i\frac{\delta\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle}{\delta f(x)} &= \langle 0\,|\,\boldsymbol{T}\,\overline{\psi}(x)\,|\,\Psi\rangle. \end{aligned}\right\} \qquad (2.9)$$

From (2.8) and the variational equations (1.14) we get for the variation of, for instance, the right-hand side of the second equation (2.9)

$$\delta\langle 0\,|\,\boldsymbol{T}\,\psi(x)\,|\,\Psi\rangle =$$

$$-i\left[\int_x^\infty \delta I(x')\langle 0\,|\,\boldsymbol{T}u(x')\,\psi(x)\,|\,\Psi\rangle + \int_{-\infty}^x \delta I(x')\langle 0\,|\,\boldsymbol{T}\psi(x)\,u(x')\,|\,\Psi\rangle\right]$$

$$-i\left[\int_x^\infty \delta g(x')\langle 0\,|\,\boldsymbol{T}\psi(x')\,\psi(x)\,|\,\Psi\rangle - \int_{-\infty}^x \delta g(x')\langle 0\,|\,\boldsymbol{T}\psi(x)\,\psi(x')\,|\,\Psi\rangle\right]$$

$$+i\left[\int_x^\infty \delta f(x')\langle 0\,|\,\boldsymbol{T}\overline{\psi}(x')\,\psi(x)\,|\,\Psi\rangle - \int_{-\infty}^x \delta f(x')\langle 0\,|\,\boldsymbol{T}\psi(x)\overline{\psi}(x')\,|\,\Psi\rangle\right],$$

whence

$$\left.\begin{aligned} i\frac{\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle}{\delta I(x')} &= \langle 0\,|\,\boldsymbol{T}\;T\big(u(x')\,\psi(x)\big)\,|\,\Psi\rangle \\ i\frac{\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle}{\delta g(x')} &= \langle 0\,|\,\boldsymbol{T}\;T\big(\psi(x')\,\psi(x)\big)\,|\,\Psi\rangle \\ -i\frac{\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle}{\delta f(x')} &= \langle 0\,|\,\boldsymbol{T}\;T\big(\overline{\psi}(x')\,\psi(x)\big)\,|\,\Psi\rangle, \end{aligned}\right\} \qquad (2.10)$$

where $T(\cdots)$ designates Wick's time-ordered product. The expressions (2.9) and (2.10) are special cases of the general formula

$$\left.\begin{aligned} & i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta I(x^{(k)})}\,i\frac{\delta}{\delta g(y')}\cdots i\frac{\delta}{\delta g(y^{(l)})} \\ & \times\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\left(-i\frac{\delta}{\delta f(z^{(m)})}\right)\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle = \\ & \langle 0|\boldsymbol{T}\;T\big(u(x')\cdots u(x^{(k)})\,\psi(y')\cdots\psi(y^{(l)})\,\overline{\psi}(z')\cdots\overline{\psi}(z^{(m)})\big)|\Psi\rangle, \end{aligned}\right\} \qquad (2.11)$$

which reveals $\boldsymbol{T}$ as the generator of the T-product. To prove (2.11) denote, for fixed values of the space time points, $x' \cdots x^{(k)}$ $y' \cdots y^{(l)}$ $z' \cdots z^{(m)}$, the chronologically ordered sequence of the same points by $x_1, x_2, \cdots x_n$, $n = k + l + m$. Further, let χ denote any of the field variables $\psi, \overline{\psi}$, and u. With this notation, we have

$$
\begin{aligned}
& \delta \langle 0 \,|\, \boldsymbol{T} \chi(x_1) \cdots \chi(x_n) \,|\, \Psi \rangle = \\
& \quad - i \langle 0 \,|\, \boldsymbol{T} \int_{x_1}^{\infty} \delta W(x)\, dx\, \chi(x_1) \cdots \chi(x_n) \,|\, \Psi \rangle, \\
& \quad - i \langle 0 \,|\, \boldsymbol{T} \chi(x_1) \int_{x_2}^{x_1} \delta W(x)\, dx \cdots \chi(x_n) \,|\, \Psi \rangle, \\
& \quad - i \cdots \\
& \quad - i \langle 0 \,|\, \boldsymbol{T} \chi(x_1) \cdots \chi(x_n) \int_{-\infty}^{x_n} \delta W(x)\, dx \,|\, \Psi \rangle,
\end{aligned}
$$

in virtue of (2.1) and the variational equations (1.14). If we displace all source variations to the extreme left we get

$$
\left.
\begin{aligned}
& \delta \langle 0 \,|\, \boldsymbol{T} \chi(x_1) \cdots \chi(x_n) \,|\, \Psi \rangle = \\
& - i \int_{-\infty}^{\infty} \delta g(x)\, (\pm) \langle 0 \,|\, \boldsymbol{T} P \{ \psi(x) \chi(x_1) \cdots \chi(x_n) \} \,|\, \Psi \rangle\, dx, \\
& + i \int_{-\infty}^{\infty} \delta f(x)\, (\pm) \langle 0 \,|\, \boldsymbol{T} P \{ \overline{\psi}(x) \chi(x_1) \cdots \chi(x_n) \} \,|\, \Psi \rangle\, dx, \\
& - i \int_{-\infty}^{\infty} \delta I(x) \qquad \langle 0 \,|\, \boldsymbol{T} P \{ u(x) \chi(x_1) \cdots \chi(x_n) \} \,|\, \Psi \rangle\, dx,
\end{aligned}
\right\} \quad (2.12)
$$

where P is Dyson's chronologically ordering operator. The $(\pm)$ factor in the two first terms on the right-hand side of (2.12) is the parity of the number of permutations between the nucleon operators and the variations of the spinor field sources. Evidently, the number of these permutations equals the number of permutations of spinor fields required to bring the field variables ψ and $\overline{\psi}$, respectively, from the place indicated in (2.12) to the position required by the P-operator. Thus, $(\pm)$ is the change of sign characterizing Wick's T-product as compared with Dyson's P-product and, hence,

$$\left.\begin{aligned}
& i\,\delta\,\langle 0 \mid \boldsymbol{T}\, T\left(u\,(x')\cdots\overline{\psi}\,(z^{(m)})\right) \mid \Psi\rangle = \\
& \int_{-\infty}^{\infty}\delta g\,(x)\,\langle 0 \mid \boldsymbol{T}\, T\left(\psi\,(x)\,u\,(x')\cdots\overline{\psi}\,(z^{(m)})\right) \mid \Psi\rangle\,dx, \\
& -\int_{-\infty}^{\infty}\delta f\,(x)\,\langle 0 \mid \boldsymbol{T}\, T\left(\overline{\psi}\,(x)\,u\,(x')\cdots\overline{\psi}\,(z^{(m)})\right) \mid \Psi\rangle\,dx, \\
& +\int_{-\infty}^{\infty}\delta I\,(x)\,\langle 0 \mid \boldsymbol{T}\, T\left(u\,(x)\,u\,(x')\cdots\overline{\psi}\,(z^{(m)})\right) \mid \Psi\rangle\,dx.
\end{aligned}\right\}\qquad (2.13)$$

The minus sign is due to the occurrence of $\delta\varphi$ to the right of $\overline{\psi}$ in the expression for δW. The proof of (2.11) is now easily completed by an induction argument.

ii) *Matrix elements of normal products.*

The formula (2.11) demonstrates the convenience of Schwinger's formalism for the introduction of ordered products of field operators, but adds nothing new. The normal product*, however, is not defined for non-linear fields and it is, therefore, more interesting that we by this formalism can give a general definition of the normal product. The detailed discussion of the normal product as introduced here, and in particular the proof that this product is a generalization of that introduced by Wick for free fields, will be given in the following section.

The generator for the N-product is the operator $\boldsymbol{N}$ which is connected with the $\boldsymbol{T}$-operator by

$$\boldsymbol{N} = \langle 0 \mid \boldsymbol{T} \mid 0\rangle^{-1}\,\boldsymbol{T}. \qquad (2.14)$$

We shall regard

$$\left.\begin{aligned}
& \Psi(x'\cdots x^{(k)} \mid y'\cdots y^{(l)} \mid z'\cdots z^{(m)}) = i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta I(x^{(k)})} \\
& i\frac{\delta}{\delta g\,(y')}\cdots i\frac{\delta}{\delta g\,(y^{(l)})}\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\left(-i\frac{\delta}{\delta f(z^{(m)})}\right)\langle 0 \mid \boldsymbol{N} \mid \Psi\rangle
\end{aligned}\right\}\qquad (2.15)$$

as the matrix element between the states $\langle 0|$ and $|\Psi\rangle$ of the N-ordered product of the field variables corresponding to the

* By Wick [9] denoted as the S-product. To avoid the use of the letter S for too many purposes we shall, henceforward, use the term N-product.

space-time points indicated[1]. The relation (2.14) implies

$$\langle 0 \mid \boldsymbol{N} \mid \Psi \rangle = \langle 0 \mid \boldsymbol{T} \mid 0 \rangle^{-1} \langle 0 \mid \boldsymbol{T} \mid \Psi \rangle \tag{2.16}$$

and

$$\langle 0 \mid \boldsymbol{T} \mid 0 \rangle \langle 0 \mid \boldsymbol{N} \mid \Psi \rangle = \langle 0 \mid \boldsymbol{T} \mid \Psi \rangle . \tag{2.17}$$

From these two expressions two relations originate between the matrix elements of N-products and the matrix elements of T-products. To express these relations in a compact form we introduce some conventions about notation.

Let $\xi', \xi'', \cdots \xi^{(\varkappa)}$, $\varkappa \leq k$ denote some of the space-time points $x', x'', \cdots x^{(k)}$. By

$$x', x'', \cdots x^{(k)}; \xi', \xi'', \cdots \xi^{(\varkappa)} \tag{2.18}$$

we denote the sequence of space-time points obtained by omitting the space-time points $\xi', \xi'', \cdots \xi^{(\varkappa)}$ from the sequence $x', x'', \cdots x^{(k)}$. Thus, for example, $x', x'', x''', x''''; x'', x''' = x', x''''$. In the same way, we introduce symbols such as $y', y'', \cdots y^{(l)}; \eta', \eta'', \cdots \eta^{(\lambda)}$ and $z', z'', \cdots z^{(m)}; \zeta', \zeta'', \cdots \zeta^{(\mu)}$.

We also introduce a notation for matrix elements of T-products similar to that we use for matrix elements of N-products. For instance, we write the right-hand side of (2.11) as

$$T_{\Psi}(x' \cdots x^{(k)} \mid y' \cdots y^{(l)} \mid z' \cdots z^{(m)}) . \tag{2.19}$$

If $\mid \Psi \rangle$ is the vacuum state, we denote the vacuum expectation value of the T-product by

$$T_0\,(x' \cdots x^{(k)} \mid y' \cdots y^{(l)} \mid z' \cdots z^{(m)}) . \tag{2.20}$$

For completeness, we note that, in the special case $k = l = m = 0$, we write

$$\left.\begin{aligned} \langle 0 \mid \boldsymbol{T} \mid \Psi \rangle &= T_{\Psi}(\mid \mid), \\ \langle 0 \mid \boldsymbol{N} \mid \Psi \rangle &= \Psi(\mid \mid). \end{aligned}\right\} \tag{2.21}$$

[1] It is evident how to generalize this definition and the formula (2.11) to matrix elements of the N- and T-products, respectively, between any two (source-independent) states of the system.

Also, with the notation (2.20), we have

$$\langle 0 \,|\, \boldsymbol{T} \,|\, 0 \rangle = T_0 (|\,|). \tag{2.22}$$

As mentioned in Section 1, matrix elements of field operators are in general not c-numbers. This introduces some minor complications in the following considerations, but is the price we have to pay in order that all three kinds of sources appear in a symmetric manner in the variational equations (1.14).

Still, any T_0-function with an even total number of spinor space-time points is effectively a c-number in the theory. Any f-number commutes with such "even" T_0-functions. The general relation for, for instance, δf is

$$\left.\begin{aligned} &\delta f\, T_0 (x' \cdots | y' \cdots y^{(l)} | z' \cdots z^{(m)}) \\ &= (-1)^{l+m}\, T_0 (x' \cdots | y' \cdots y^{(l)} | z' \cdots z^{(m)})\, \delta f, \end{aligned}\right\} \tag{2.23}$$

and is easily proved by the use of (1.16) and the anti-commutativity of $\delta\varphi$ with all spinor fields. A similar relation holds for δg, f, and g. Hence, even T_0-functions commute with any functional of f and g and, in particular, with any other T_0-function. Thus,

$$\left[T_0 (x' \cdots | y' \cdots y^{(l)} | z' \cdots z^{(m)}),\ T_0 (x_1 \cdots | y_1 \cdots y_\lambda | z_1 \cdots z_\mu)\right] = 0$$

if $\lambda + \mu$ is even. The variational derivative of this equation with respect to $g(y)$ gives, for the case of $l + m$ being odd,

$$\begin{aligned} &\left[T_0 (x' \cdots | y y' \cdots y^{(l)} | z' \cdots z^{(m)}),\ T_0 (x_1 \cdots | y_1 \cdots y_\lambda | z_1 \cdots z_\mu)\right] \\ &-\left\{T_0(x' \cdots | y' \cdots y^{(l)} | z' \cdots z^{(m)}), T_0(x_1 \cdots | y y_1 \cdots y_\lambda | z_1 \cdots z_\mu)\right\} = 0. \end{aligned}$$

The appearance of an anti-commutator is a consequence of the anti-commutativity of δg with odd T_0-functions. The first term vanishes and we infer that two T_0-functions, both having an odd total number of spinor space-time points, anti-commute. In particular, $T_0 (|\,|)$ commutes with all matrix elements and

$$\left.\begin{aligned}\left\{T_0(|y|),\ T_0(|y'|)\right\} &= 0,\\ \left\{T_0(|y|),\ T_0(||z)\right\} &= 0,\\ \left\{T_0(||z),\ T_0(||z')\right\} &= 0.\end{aligned}\right\} \qquad (2.24)$$

We are now prepared to prove the first of the relations mentioned above. From (2.17) follows

$$\left.\begin{aligned} &T_{\Psi}(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)})\\ &= \sum_{\varkappa\lambda\mu}\frac{1}{\varkappa!}\sum_{\xi'\cdots\xi^{(\varkappa)}}\frac{1}{\lambda!}\sum_{\eta'\cdots\eta^{(\lambda)}}\frac{1}{\mu!}\sum_{\zeta'\cdots\zeta^{(\mu)}}\cdot\\ &\times(\pm)\,T_0(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)})\\ \Psi\,(x'\cdots x^{(k)};\ \xi'\cdots\xi^{(\varkappa)}\,|\,&y'\cdots y^{(l)};\ \eta'\cdots\eta^{(\lambda)}\,|\,z'\cdots z^{(m)};\ \xi'\cdots\xi^{(\mu)}),\end{aligned}\right\} \qquad (2.25)$$

where the summation is taken over $\varkappa = 0, 1, \cdots k$, $\lambda = 0, 1, \cdots l$, and $\mu = 0, 1, \cdots m$ while the ξ's run independently over all the space-time points $x' \cdots x^{(k)}$, etc. The factorials take into account that we sum over all permutations of the sets $\xi' \cdots$, $\eta' \cdots$ and $\zeta' \cdots$. Apart from the factor $(\pm)$ in front of the general term, (2.25) is easily recognized as the usual formula for the iterated derivative of a product, viz. the product on the left-hand side of (2.17). Thus, (2.25) is correct if we interpret the sign factor $(\pm)$ correctly. From (2.23) it follows, however, that the factor $(\pm)$ is the parity of the permutation of spinor space-time points involved in the substitution

$$\left.\begin{aligned} &(x'x''\cdots x^{(k)}|\,y'y''\cdots y^{(l)}|\,z'z''\cdots z^{(m)}) \to\\ &\qquad(\xi'\cdots\xi^{(\varkappa)}|\,\eta'\cdots\eta^{(\lambda)}|\,\zeta'\cdots\zeta^{(\mu)})\\ (x'\cdots x^{(k)};\ \xi'\cdots\xi^{(\varkappa)}|\,&y'\cdots y^{(l)};\ \eta'\cdots\eta^{(\lambda)}|\,z'\cdots z^{(m)};\ \zeta'\cdots\zeta^{(\mu)}).\end{aligned}\right\} \qquad (2.26)$$

To illustrate (2.25) we note a few examples which also later will serve for reference:

$$\left.\begin{aligned} T_{\Psi}(x'x''|\,|) = T_0(|\,|)\,\Psi(x'x''|\,|) + T_0(x'|\,|)\,\Psi(x''|\,|) \\ + T_0(x''|\,|)\,\Psi(x'|\,|) + T_0(x'x''|\,|)\,\Psi(|\,|) \end{aligned}\right\} \quad (2.27)$$

$$\left.\begin{aligned} T_{\Psi}(x\,|y|) = T_0(|\,|)\,\Psi(x\,|y|) + T_0(x\,|\,|)\,\Psi(|y|) \\ + T_0(|y|)\,\Psi(x|\,|) + T_0(x\,|y|)\,\Psi(|\,|) \end{aligned}\right\} \quad (2.28)$$

$$\left.\begin{aligned} T_{\Psi}(x|\,|z) = T_0(|\,|)\,\Psi(x|\,|z) + T_0(x|\,|)\,\Psi(|\,|z) \\ + T_0(|\,|z)\,\Psi(x|\,|) + T_0(x|\,|z)\,\Psi(|\,|), \end{aligned}\right\} \quad (2.29)$$

and, finally, to illustrate the $(\pm)$ factor,

$$\left.\begin{aligned} T_{\Psi}(|y'y''|) = T_0(|\,|)\,\Psi(|y'y''|) + T_0(|y'|)\,\Psi(|y''|) \\ - T_0(|y''|)\,\Psi(|y'|) + T_0(|y'y''|)\,\Psi(|\,|). \end{aligned}\right\} \quad (2.30)$$

The formula (2.25) may be looked upon as a recursion formula which implicitly expresses the Ψ-functions in terms of matrix elements of T-products. The resulting formula may, however, be obtained directly from (2.16) if we introduce the functions

$$\left.\begin{aligned} C\,(x'\cdots x^{(k)}|y'\cdots y^{(l)}|z'\cdots z^{(m)}) = i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta I(x^{(k)})} \\ i\frac{\delta}{\delta g(y')}\cdots i\frac{\delta}{\delta g(y^{(l)})}\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\left(-\frac{\delta}{\delta f(z^{(m)})}\right)C(|\,|), \end{aligned}\right\} \quad (2.31)$$

where

$$C(|\,|) = \langle 0\,|\,\boldsymbol{T}\,|0\rangle^{-1}. \quad (2.32)$$

By an argument similar to that by which (2.25) was obtained, we get from (2.16)

$$\left.\begin{aligned} &\Psi\,(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}) \\ &= \sum_{\varkappa\lambda\mu}\frac{1}{\varkappa!}\sum_{\xi'\cdots\xi^{(\varkappa)}}\frac{1}{\lambda!}\sum_{\eta'\cdots\eta^{(\lambda)}}\frac{1}{\mu!}\sum_{\zeta'\cdots\zeta^{(\mu)}} \\ &(\pm)\,C\,(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}) \\ &T_{\Psi}\,(x'\cdots x^{(k)};\,\xi'\cdots\xi^{(\varkappa)}\,|\,y'\cdots y^{(l)};\,\eta'\cdots\eta^{(\lambda)}\,|z'\cdots z^{(m)};\,\zeta'\cdots\zeta^{(\mu)}). \end{aligned}\right\} \quad (2.33)$$

An important property of the Ψ-functions follows from (2.14). In the special case where $|\Psi\rangle$ is the vacuum state of the source-free system, we have $\langle 0 | \boldsymbol{N} | 0 \rangle = 1$, independent of the sources. Hence, all Ψ-functions vanish, except the one corresponding to $k = l = m = 0$. Thus, in this case, (2.33) reduces to

$$\left.\begin{aligned} &\sum_{\varkappa\lambda\mu}{}' \frac{1}{\varkappa!} \sum_{\xi'\cdots\xi^{(\varkappa)}}{}' \frac{1}{\lambda!} \sum_{\eta'\cdots\eta^{(\lambda)}}{}' \frac{1}{\mu!} \sum_{\zeta'\cdots\zeta^{(\mu)}}{}' \\ &\qquad (\pm)\, C\,(\xi'\cdots\xi^{(\varkappa)} | \eta'\cdots\eta^{(\lambda)} | \zeta'\cdots\zeta^{(\mu)}) \\ T_0\,(x'\cdots x^{(k)};\, &\xi'\cdots\xi^{(\varkappa)} | y'\cdots y^{(l)};\, \eta'\cdots\eta^{(\lambda)} | z'\cdots z^{(m)};\, \zeta'\cdots\zeta^{(\mu)}) \\ &\qquad = \delta_{0k}\delta_{0l}\delta_{0m}, \end{aligned}\right\} \tag{2.34}$$

and this is a recursion formula expressing the C-functions in terms of vacuum expectation values of T-products[1].

In the following, the formulas (2.25), (2.33), and (2.34) will serve as a basis for the discussion of the properties of the matrix elements of N-products. It will be shown that these expressions generalize the algebraic relations between T- and N-products for free fields to the case of non-linear fields.

3. Properties of matrix elements of *N*-products.

In the Fock representation [7] in configuration space for free fields, one characterizes a state of the system by an infinite set of many-particle wave functions. As long as one considers free fields, this representation may in a trivial way be extended to a multiple time representation. If we use the notion of a normal product, we can write the many-time wave functions, or as we prefer to say here, the state vector amplitudes, in the form

$$\left.\begin{aligned} &\Psi\,(x'\cdots x^{(k)} | y'\cdots y^{(l)} | z'\cdots z^{(m)}) \\ &= \langle 0 | N(u(x')\cdots u(x^{(k)})\psi(y')\cdots\psi(y^{(l)})\bar{\psi}(z')\cdots\bar{\psi}(z^{(m)})) | \Psi\rangle. \end{aligned}\right\} \tag{3.1}$$

The results of Wick's discussion of the properties of T- and N-products of free field operators are expressed in the Appendix

[1] The formula (2.34) could also have been obtained directly from the identity $C\,(||)\,T_0\,(||) = 1$.

II with the aid of a notation which is convenient for our purpose. In Section 2, we derive relations connecting matrix elements of the normal product of field operators for non-linear fields with matrix elements of T-products. If we compare the formula (2.25) with Wick's formula (Ap. II. 11), we see that the N-product for non-linear fields, as defined by (2.15), is a generalization of the N-product for free fields, as the formula (2.25) in the limit $\varphi = \overline{\varphi} = I = \lambda = 0$ reduces to the corresponding formula for free fields given in the Appendix II.

The equation (2.15) may, therefore, be taken as the general definition, valid also for non-linear fields, of the state vector amplitudes which represent any given state $|\Psi\rangle$. After a discussion, in this section, of some of the simplest properties of the state vector amplitudes, we shall in the following section derive the equations of motion in this new representation. It will then be seen that the state vector amplitudes are closely connected to the "wave functions" which enter in the homogeneous equations of motion following from Schwinger's theory.

The following simple properties of the state vector amplitudes are independent of the magnitude of the coupling constant.

i) The ground state of the source-free system has the representation $\Psi(||) = 1$, while all other amplitudes vanish. As already remarked at the end of the last section, this follows in a trivial way from the definitions (2.15) and (2.14). The fact that the simplest state of the system has the simplest possible representation is in accordance with the expectation that the present formalism provides us with a convenient description of the lowest lying levels of the system.

ii) It is easily verified from (2.25), by means of well-known properties of T-products, that the state vector amplitudes are symmetric functions in all meson coordinates and anti-symmetric in as well all nucleon coordinates as all anti-nucleon coordinates. So far we have not introduced Ψ-functions such that we can speak about symmetry properties when interchanges of, for instance, nucleons and anti-nucleons are involved. It is, however, evident how one could generalize (2.15) to cover such cases also. One would then obtain state vector amplitudes which, in the general case, possess all the well-known symmetry properties of the free field wave function (3.1). The most direct way to see this is to observe that we formally can use the relations

$$\left.\begin{aligned}&\left\{i\frac{\delta}{\delta g(y')},\, i\frac{\delta}{\delta g(y'')}\right\}=0,\\&\left\{i\frac{\delta}{\delta g(y')},\, -i\frac{\delta}{\delta f(z')}\right\}=0,\\&\left\{-i\frac{\delta}{\delta f(z')},\, -i\frac{\delta}{\delta f(z'')}\right\}=0,\end{aligned}\right\}\tag{3.2}$$

and commutativity of $i\delta/\delta I$ with all variational derivative operators when the objects of operation are matrix elements of T- and N-products. To this remark we shall come back in the next section. To illustrate (3.2) we evaluate

$$\left.\begin{aligned}&\left(-i\frac{\delta}{\delta f(z)}\right)\langle 0\,|\,\boldsymbol{T}\,T(\psi(y')\cdots\psi(y^{(l)}))\,|\,\Psi\rangle\\&=\langle 0\,|\,\boldsymbol{T}\,T(\bar{\psi}(z)\,\psi(y')\cdots\psi(y^{(l)}))\,|\,\Psi\,\rangle\\&=(-1)^l\,T_{\Psi}(|\,y'\cdots y^{(l)}\,|\,z).\end{aligned}\right\}\tag{3.3}$$

Here we have used (2.13) and the symmetry properties of T-products.

iii) The state vector amplitudes are continuous functions of the coordinates. This is not quite trivial, because matrix elements of T-products are, in general, discontinuous functions. The discontinuous character of the T_{Ψ}-functions is made apparent by the δ-terms in the equations of motion for these functions (Ap. III. 3, 4 and 5). It can, however, be seen that the application of the differential operators occurring in the field equations to Ψ-functions does not give rise to such δ-functions. This can, for instance, be proved by induction using (2.25). In the following section, we find that the Ψ-functions satisfy homogeneous equations of motion, and this constitutes another verification of the continuity of these functions[1].

[1] The first derivative of the Ψ-functions with respect to a meson coordinate is also continuous. This difference between spinor field variables and meson field variables reflects the difference in the equations of motion for the two kinds of fields, the nucleon equations being of the first order, while the meson equation is of the second order.

iv) If the state $|\Psi\rangle$ represented by the infinite set of state vector amplitudes is a stationary state, corresponding to the eigenvalues P_μ for the total energy momentum vector of the closed system, then, in the source-free limit, the Ψ-functions oscillate according to

$$\Psi(x' \cdots | y' \cdots | z' \cdots) \sim \exp i P_\mu X_\mu. \qquad (3.4)$$

Here, the X_μ's are any "center of gravity" coordinate. For instance, one can take X_μ as the average value of the coordinates $x'_\mu, \cdots y'_\mu, \cdots z'_\mu, \cdots$.[1] This follows immediately from (2.33) and the fact that T_Ψ-functions possess this property. The property (3.4) is of course the basis for the application of the present formalism to bound state problems.

v) The configuration space representation. The state vector amplitudes corresponding to a state $|\Psi\rangle$ provide us with a generalization of the Fock representation for free fields. As we have seen above, several of the simple properties of the Fock representation are maintained in the general case. One might, therefore, consider the set of state vector amplitudes as a representation of the state $|\Psi\rangle$. We shall take such a point of view in the following, and speak of this representation as the configuration space representation. Alternatively, we can also consider the functional $\Psi(||)$ which generates the state vector amplitudes as representing the state in question. In this way we speak of the functional representation. For the sake of convenience, we denote these two representations by the CSR and the FR, respectively.

To make full use of the CSR one should know, at least in principle, how to construct the scalar product of two states represented by their state vector amplitudes. This problem could not be solved and we have not even been able to prove that the CSR is a complete representation. Until further progress is

[1] Cf. Freese [4]. As mentioned by Freese, the most general definition of X_μ is

$$X_\mu = \alpha' x'_\mu + \cdots + \beta' y'_\mu + \cdots + \gamma' z'_\mu + \cdots,$$

where the α, β, and γ's are subject to the condition

$$\alpha' + \cdots + \beta' + \cdots + \gamma' + \cdots = 1.$$

made, application of the present formalism must therefore be based on an assumption of the completeness of the representation.

A comparison of the CSR with other better known representations might offer a possibility of discussing the completeness problem. The fact that $\langle 0 |$ is the vacuum state of the source-free system has been used in the discussion of the oscillating behaviour of the amplitudes representing stationary states. It is easily seen that all other considerations remain valid for any choice of $\langle 0 |$ if only this state coincides with the free-field vacuum in the limit of no coupling. An example of another possible choice of this state is provided by the vacuum state $\langle 0, \sigma |$ for the free fields $u(x, \sigma)$, $\psi(x, \sigma)$ which coincide with the source independent fields on a space-like surface σ. Moreover, it can be seen that one can choose different states in the definition of the functional $\Psi(||)$. Thus, instead of (2.15), we could have defined

$$\Psi_{\sigma'}^{\sigma''}(||) = \frac{\langle 0, \sigma'' | \boldsymbol{T} | \Psi \rangle}{\langle 0, \sigma' | \boldsymbol{T} | 0, \sigma' \rangle},$$

where $|0, \sigma''\rangle$ and $|0, \sigma'\rangle$ may be different.

The choice

$$\Psi_{\sigma}^{\sigma}(||) = \frac{\langle 0, \sigma | \boldsymbol{T} | \Psi \rangle}{\langle 0, \sigma | \boldsymbol{T} | 0, \sigma \rangle} \tag{3.5}$$

leads to a representation in which the state vector amplitudes for all space-time points on σ coincide with the Tamm-Dancoff representation.

As is well known, one can consider the state $|0\rangle$ as the limit of $|0, \sigma\rangle$ in the sense of a certain limiting process, usually referred to as the adiabatic switching-on of the coupling at $t = -\infty$. In the sense of the same limiting process, one can regard the CSR representation as the limit of the representation based on (3.5) for $\sigma \to -\infty$. The coincidence of the representation (3.5) with the Tamm-Dancoff representation on σ tells us that (3.5) is a complete representation. There might, therefore, be a possibility of discussing the completeness of the CSR by a comparison with the Tamm-Dancoff representation. The complexity of the limiting process involved, however, does not make this a very promising prospect.

Another representation could be based on the functional

$$\Psi_\sigma(||) = \frac{\langle 0 | \boldsymbol{T} | \Psi \rangle}{\langle 0, \sigma | \boldsymbol{T} | 0, \sigma \rangle}. \tag{3.6}$$

The corresponding state vectors can be seen to coincide on σ with the representation given by DYSON [12].

4. The equations of motion.

In the preceding section, we have introduced two new representations, the functional representation and the configuration space representation. The simplest way to obtain the equations of motion in these two representations is first to derive the equations of motion in the FR. As we shall see, the equations of motion in the CSR can be obtained from those in the FR by a simple procedure.

i) The equations of motion in the FR. To determine the dependence of the functional Ψ on the sources we must try to set up variational equations making use of the field equations. The Ψ-functions depending on one space-time point only are given by the expressions

$$\left.\begin{aligned}
\Psi(x||) &= T_0(||)^{-1} T_{\Psi}(x||) - T_0(||)^{-1} T_0(x||)\, T_0(||)^{-1} T_{\Psi}(||),\\
\Psi(|y|) &= T_0(||)^{-1} T_{\Psi}(|y|) - T_0(||)^{-1} T_0(|y|)\, T_0(||)^{-1} T_{\Psi}(||),\\
\Psi(||z) &= T_0(||)^{-1} T_{\Psi}(||z) - T_0(||)^{-1} T_0(||z)\, T_0(||)^{-1} T_{\Psi}(||).
\end{aligned}\right\} \tag{4.1}$$

These equations are special cases of the formula (2.33), but can also easily be verified directly from the definition (2.15). As shown in the Appendix II, the T_{Ψ}-functions depending on one space-time point satisfy

$$\left.\begin{aligned}
(-\Box_x + m^2)\, T_{\Psi}(x||) - \lambda\, T_{\Psi}(|x|x) + I(x)\, T_{\Psi}(||) &= 0,\\
(\vartheta_y + M)\, T_{\Psi}(|y|) + \lambda\, T_{\Psi}(y|y|) + f(y)\, T_{\Psi}(||) &= 0,\\
(\bar{\vartheta}_z + M)\, T_{\Psi}(||z) + \lambda\, T_{\Psi}(z||z) + g(z)\, T_{\Psi}(||) &= 0.
\end{aligned}\right\} \tag{4.2}$$

These equations are of course also satisfied in the special case of $|\Psi\rangle$ being the vacuum state, i. e. for T_0-functions. Combining (4.1) and (4.2) we get

$$\left.\begin{aligned} &(-\Box_x + m^2)\, \Psi(x\,|\,|) - \lambda\, T_0(|\,|)^{-1}\, T_{\Psi}(|\,x\,|\,x) \\ &\quad + \lambda\, T_0(|\,|)^{-1}\, T_0(|\,x\,|\,x)\, \Psi(|\,|) = 0, \\ &(\partial_y + M)\, \Psi(|\,y\,|) + \lambda\, T_0(|\,|)^{-1}\, T_{\Psi}(y\,|\,y\,|) \\ &\quad - \lambda\, T_0(|\,|)^{-1}\, T_0(y\,|\,y\,|)\, \Psi(|\,|) = 0, \\ &(\bar{\partial}_z + M)\, \Psi(|\,|\,z) + \lambda T_0(|\,|)^{-1}\, T_{\Psi}(z\,|\,|\,z) \\ &\quad - \lambda\, T_0(|\,|)^{-1}\, T_0(z\,|\,|\,z)\, \Psi(|\,|) = 0, \end{aligned}\right\} \tag{4.3}$$

where the sources do no longer explicitly appear. To express (4.3) as linear equations in Ψ and the variational derivatives of Ψ, we eliminate the T_{Ψ}-functions by use of expressions of the type (2.28), (2.29). The resulting equations contain as factors certain combinations of T-functions for which we introduce the notation

$$\left.\begin{aligned} \eta(x\,|\,|) &= T_0(|\,|)^{-1}\, T_0(x\,|\,|), \\ \eta(|\,y\,|) &= T_0(|\,|)^{-1}\, T_0(|\,y\,|), \\ \eta(|\,|\,z) &= T_0(|\,|)^{-1}\, T_0(|\,|\,z). \end{aligned}\right\} \tag{4.4}$$

By the aid of these η-functions we can write the resulting linear differential variational equations for Ψ in the form

$$\left.\begin{aligned} &(-\Box_x + m^2)\, \Psi(x\,|\,|) - \lambda\eta(|\,x\,|)\, \Psi(|\,|\,x) \\ &\quad + \lambda\eta(|\,|\,x)\, \Psi(|\,x\,|) - \lambda\Psi(|\,x\,|\,x) = 0, \\ &(\partial_y + M)\, \Psi(|\,y\,|) + \lambda\eta(y\,|\,|)\, \Psi(|\,y\,|) \\ &\quad + \lambda\eta(|\,y\,|)\, \Psi(y\,|\,|) + \lambda\Psi(y\,|\,y\,|) = 0, \\ &(\bar{\partial}_z + M)\, \Psi(|\,|\,z) + \lambda\eta(z\,|\,|)\, \Psi(|\,|\,z) \\ &\quad + \lambda\eta(|\,|\,z)\, \Psi(z\,|\,|) + \lambda\Psi(z\,|\,|\,z) = 0. \end{aligned}\right\} \tag{4.5}$$

Thus, for any state $|\,\Psi\rangle$, the corresponding functional Ψ satisfies (4.5). The problem which restrictions (if any) must be imposed on the solutions of (4.5) to guarantee that the functional obtained represents a state of the system has not been solved in general. Hereto we shall return later.

ii) The *η-functions.* By a similar method we obtain equations of motion for the η-functions. Combining (4.2) and (4.4) we get

$$\left.\begin{aligned}(-\Box_x+m^2)\,\eta(x\,|\,|\,)-\lambda\,T_0(\,|\,|\,)^{-1}\,T_0(\,|\,x\,|\,x)+I(x)&=0,\\(\vartheta_y+M)\,\eta(\,|\,y\,|\,)+\lambda\,T_0(\,|\,|\,)^{-1}\,T_0(y\,|\,y\,|\,)+f(y)&=0,\\(\bar{\vartheta}_z+M)\,\eta(\,|\,|\,z)+\lambda\,T_0(\,|\,|\,)^{-1}\,T_0(z\,|\,|\,z)+g(z)&=0.\end{aligned}\right\}\quad(4.6)$$

It is convenient to introduce a functional η by

$$\eta(\,|\,|\,)=\log\langle 0\,|\,\boldsymbol{T}\,|\,0\rangle.\quad(4.7)$$

The η-functions (4.4) are contained as special cases in the following general definition of η-functions:

$$\left.\begin{aligned}&\eta(x'\cdots|\,y'\cdots|\,z'\cdots)\\&=i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta g(y')}\cdots\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\eta(\,|\,|\,).\end{aligned}\right\}\quad(4.8)$$

Thus, for instance, η-functions depending on two space-time points are given by

$$\left.\begin{aligned}\eta(\,|\,y\,|\,z)&=T_0(\,|\,|\,)^{-1}\,T_0(\,|\,y\,|\,z)-\eta(\,|\,y\,|\,)\,\eta(\,|\,|\,z),\\\eta(x\,|\,y\,|\,)&=T_0(\,|\,|\,)^{-1}\,T_0(x\,|\,y\,|\,)-\eta(x\,|\,|\,)\,\eta(\,|\,y\,|\,),\\\eta(x\,|\,|\,z)&=T_0(\,|\,|\,)^{-1}\,T_0(x\,|\,|\,z)-\eta(x\,|\,|\,)\,\eta(\,|\,|\,z).\end{aligned}\right\}\quad(4.9)$$

With the aid of these formulas we can eliminate the T_0-functions in (4.6) and obtain

$$\left.\begin{aligned}(-\Box_x+m^2)\,\eta(x\,|\,|\,)-\lambda\eta(\,|\,x\,|\,)\,\eta(\,|\,|\,x)-\lambda\eta(\,|\,x\,|\,x)+I(x)&=0,\\(\vartheta_y+M)\,\eta(\,|\,y\,|\,)+\lambda\eta(y\,|\,|\,)\,\eta(\,|\,y\,|\,)+\lambda\eta(y\,|\,y\,|\,)+f(y)&=0,\\(\bar{\vartheta}_z+M)\,\eta(\,|\,|\,z)+\lambda\eta(z\,|\,|\,)\,\eta(\,|\,|\,z)+\lambda\eta(z\,|\,|\,z)+g(z)&=0.\end{aligned}\right\}\quad(4.10)$$

These equations are variational differential equations satisfied by the η-functional in the FR.

Contrary to the Ψ-functional which depends on the particular state considered, the η-functional is uniquely determined in the theory. We must, therefore, supplement the η-equations by boundary conditions characterizing the particular solution of (4.10) which enters in the equations of motion for the Ψ-functional.

For the discussion of this problem we need an interpretation of the operator $\boldsymbol{T}$. Let $\boldsymbol{T}(t)$ be the transformation which connects the source-free fields and the source-dependent fields according to

$$\left.\begin{aligned} u(x) &= \boldsymbol{T}(t)^{-1}\, u_0(x)\, \boldsymbol{T}(t), \\ \psi(x) &= \boldsymbol{T}(t)^{-1}\, \psi_0(x)\, \boldsymbol{T}(t). \end{aligned}\right\} \tag{4.11}$$

As may be seen from (1.14), $\boldsymbol{T}(t)$ satisfies the variational equation

$$\delta\, \boldsymbol{T}(t) = -\, i\, \boldsymbol{T}(t) \int_{-\infty}^{t} \delta\, W(x')\, dx', \tag{4.12}$$

and the boundary condition $\boldsymbol{T}(t) = 1$ in the limit of vanishing sources. Hence, we see that the operator $\boldsymbol{T}$, as defined by (2.1), can be interpreted as the transformation which connects the source-independent fields with the complete source-dependent fields in the infinite future, i. e.

$$\lim_{t \to \infty} (\boldsymbol{T}^{-1}\, u_0(x)\, \boldsymbol{T} - u(x)) = 0,$$

and similar relations for the two other fields. We shall use these relations in the form

$$\left.\begin{aligned} &\lim_{t \to \infty} (u_0(x)\, \boldsymbol{T} - \boldsymbol{T}\, u(x)) = 0, \\ &\lim_{t \to \infty} (\psi_0(x)\, \boldsymbol{T} - \boldsymbol{T}\, \psi(x)) = 0, \\ &\lim_{t \to \infty} (\overline{\psi}_0(x)\, \boldsymbol{T} - \boldsymbol{T}\, \overline{\psi}(x)) = 0. \end{aligned}\right\} \tag{4.13}$$

Assume now, as we already tacitly have done in the previous considerations, that the source-independent system by a suitable renormalization has been cast into a form such that a state of lowest energy, the vacuum state, exists and that the energy and momentum of this state is zero. It follows that any stationary state of the system corresponds to an energy momentum vector lying inside the half cone in momentum space characterized by $p_\mu p_\mu < 0$ and $p_0 > 0$. Evidently, time-like momenta corresponding to negative energy are excluded by the assumption made. However, also space-like momenta are excluded since, by a suitable Lorentz transformation, any space-like momentum vector might be brought into a form with $p_0 < 0$, i. e. with negative energy. Corresponding to the invariant decomposition of momentum space into the three subspaces: the positive frequency part (or the (+)-part) characterized by $p_\mu p_\mu < 0, p_0 > 0$, the negative frequency part (or the (—)-part) characterized by $p_\mu p_\mu < 0, p_0 < 0$, and the (0)-part with $p_\mu p_\mu > 0$, we can split any field variable into three parts, the (+)- the (—)- and the (0)-part. For instance, if we define $u(p)$ by

$$u(x) = (2\pi)^{-2} \int u(p)\, e^{ipx}\, dp, \tag{4.14}$$

we have

$$\left.\begin{aligned} u^{(+)}(x) &= (2\pi)^{-2} \int_{p^2 < 0,\, p_0 > 0} u(p)\, e^{ipx}\, dp, \\ u^{(-)}(x) &= (2\pi)^{-2} \int_{p^2 < 0,\, p_0 < 0} u(p)\, e^{ipx}\, dp, \\ u^{(0)}(x) &= (2\pi)^{-2} \int_{p^2 > 0} u(p)\, e^{ipx}\, dp. \end{aligned}\right\} \tag{4.15}$$

From our assumption it follows that

$$u_0(x)\, |0\rangle = u_0^{(-)}(x)\, |0\rangle. \tag{4.16}$$

Hence also, as $\lim\limits_{t \to -\infty} (u(x) - u_0(x)) = 0$,

$$\lim_{t \to -\infty} (u(x) - u_0^{(-)}(x))\, |0\rangle = 0, \tag{4.17}$$

i. e. $u(x)\,|0\rangle$ contains only negative frequencies in the infinite past. The corresponding statement about the asymptotic behaviour of the field variables when multiplied by $\langle 0|\,\boldsymbol{T}$ from the left follows from (4.13). By the same kinds of arguments as those leading to (4.17) we get

$$\lim_{t \to +\infty} \langle 0|\, \boldsymbol{T}\, (u(x) - u_0^{(+)}(x)) = 0. \tag{4.18}$$

Thus, $\langle 0|\,\boldsymbol{T}\, u(x)$ contains only positive frequencies in the infinite future. The same result applies to the two other kinds of field variables.

Consider any T_0-function

$$T_0\,(x' \cdots x^{(\nu)} \cdots x^{(k)}\,|\,y' \cdots |\,z' \cdots),$$

say. In the limit where one of the space-time points, for instance $x^{(\nu)}$, tends to $-\infty$, we have, considering all other space-time points as fixed,

$$\left.\begin{aligned} &\lim_{x_0^{(\nu)} \to -\infty} [T_0\,(x' \cdots x^{(\nu)} \cdots x^{(k)}|\,y' \cdots |\,z' \cdots) \\ &\quad - \langle 0\,|\,\boldsymbol{T}\,T\,(u\,(x') \cdots u\,(x^{(\nu-1)})\,u\,(x^{(\nu+1)}) \cdots u\,(x^{(k)}) \cdots) \\ &\qquad \times u\,(x^{(\nu)})\,|\,0\rangle] = 0. \end{aligned}\right\} \quad (4.19)$$

Hence, we infer from (4.17) that in the limit $x_0^{(\nu)} \to -\infty$ the T_0-function contains only negative frequencies in a Fourier decomposition with respect to $x^{(\nu)}$. The same property holds for any other space-time point occurring in a T_0-function. In the opposite limit, we get by a similar argument that T_0 contains only positive frequencies corresponding to any space-time point approaching the infinite future. Using a terminology which is suggestive in connection with the discussion, given by STUECKELBERG, FEYNMAN and FIERZ [11], of the properties of the causal Green's functions, we say that T_0-functions obey causal boundary conditions. The possibility of expressing the η-functions in terms of T_0-functions (as, for instance, expressed by (4.8) and (4.7)) implies that also η-functions satisfy causal boundary conditions.

The equations for the η-functions (4.10) are of the second order in the variational derivatives. We must therefore supplement the boundary conditions with the value of the functional η and its first variational derivative in the limit of vanishing sources. In this limit, however, $\boldsymbol{T} = 1$. Hence, $\eta\,(x\,|\,|) = \langle 0\,|\,u_0(x)\,|\,0\rangle = 0$, in virtue of (4.16). Similarly, in the same limit, $\eta\,(|\,y\,|) = \eta\,(|\,|\,z) = 0$. Finally, by the definition (4.7) we have chosen $\eta\,(|\,|)\,|_{I = \varphi = \bar{\varphi} = 0} = 0$.

Similar considerations apply to the state vector amplitudes in the infinite future. This is obvious from (2.33) or alternatively from the definition (2.15). Hence, Ψ-functions obey causal boundary conditions in the infinite future. In the infinite past, however, the behaviour of the Ψ-functions depend on the particular state considered. One more information about the Ψ-functions follows from the considerations in Section 3, iv). If we consider the Ψ-functions for all time variables equal, then, in the source-free limit, only positive frequencies are allowed with respect to this common time.

It is not known whether more conditions must be imposed on the state vector amplitudes to guarantee that a solution of the equations of motion (4.5) actually represents a state of the system. The solution of this problem is of course connected with the likewise unsolved problem of the completeness of the CSR.

iii) The equations of motion in the CSR. Having thus obtained the equations of motion in the FR it becomes a simple matter to derive the equations of motion in the CSR. As mentioned in the Appendix III, in connection with the derivation of the equations of motion of the time ordered products, the differential operators occurring in the field equations commute with all variational operators. We can, therefore, obtain an infinite set of linear differential equations for the Ψ-functions by variational derivation of the equations (4.5). For instance, by applying the variational operator $i\,\delta/\delta I\,(x')$ to the first equation (4.5), we get

$$\left.\begin{aligned}&(-\Box_x+m^2)\,\Psi(xx'\,|\,|)-\lambda\eta\,(|\,x\,|)\,\Psi(x'\,|\,|\,x)\\&+\lambda\eta\,(|\,|\,x)\,\Psi(x'\,|\,x\,|)-\lambda\eta\,(x'\,|\,x\,|)\,\Psi(|\,|\,x)\\&+\lambda\eta\,(x'\,|\,|\,x)\,\Psi(|\,x\,|)-\lambda\,\Psi(x'\,|\,x\,|\,x)=0\,.\end{aligned}\right\}\quad(4.20)$$

Similarly, from the second equation (4.5), we infer

$$\left.\begin{aligned}&(\partial_y+M)\,\Psi(x'\,|\,y\,|)+\lambda\eta\,(y\,|\,|)\,\Psi(x'\,|\,y\,|)\\&+\lambda\eta\,(|\,y\,|)\,\Psi(x'y\,|\,|)+\lambda\eta\,(yx'\,|\,|)\,\Psi(|\,y\,|)\\&+\lambda\eta\,(x'\,|\,y\,|)\,\Psi(y\,|\,|)+\lambda\,\Psi(x'y\,|\,y\,|)=0\,.\end{aligned}\right\}\quad(4.21)$$

Proceeding, and taking variational derivatives, one can construct equations involving Ψ-functions with an arbitrary number of meson space-time coordinates. Equations involving one more nucleon space time point and one more anti-nucleon space time point are obtained by applying the operators $i\delta/\delta g(y')$ and $-i\delta/\delta f(z')$, respectively. Only, when varying the spinor sources, one should remember the anti-commutativity (3.2) of the nucleon variational operators. Thus,

$$
\begin{aligned}
& i\frac{\delta}{\delta g(y')}\left(\eta(y\,|\,|)\,\Psi(|\,y\,|)\right) = \eta(y\,|\,y'\,|)\,\Psi(|\,y\,|) - \eta(y\,|\,|)\,\Psi(|\,yy'\,|), \\
& i\frac{\delta}{\delta g(y')}\left(\eta(|\,y\,|)\,\Psi(y\,|\,|)\right) = -\eta(|\,yy'\,|)\,\Psi(y\,|\,|) - \eta(|\,y\,|)\,\Psi(y\,|\,y'\,|), \\
& \qquad i\frac{\delta}{\delta g(y')}\,\Psi(y\,|\,y\,|) = -\Psi(y\,|\,yy'\,|).
\end{aligned}
$$

Observing this, we get by applying $i\delta/\delta g(y')$ to the second equation (4.5)

$$
\left.
\begin{aligned}
(\partial_y + M)\,\Psi(|\,yy'\,|) + \lambda\eta(y\,|\,|)\,\Psi(|\,yy'\,|) + \lambda\,\Psi(y\,|\,yy'\,|) \\
-\lambda\eta(y\,|\,y'\,|)\,\Psi(|\,y\,|) + \lambda\eta(|\,yy'\,|)\,\Psi(y\,|\,|) \\
+\lambda\eta(|\,y\,|)\,\Psi(y\,|\,y'\,|) = 0.
\end{aligned}
\right\} \quad (4.22)
$$

By a similar procedure one obtains equations connecting the various η-functions. For later reference we note a few examples:

$$
\left.
\begin{aligned}
(-\Box_x + m^2)\,\eta(xx'\,|\,|) - \lambda\eta(x'\,|\,x\,|)\,\eta(|\,|\,x) - \lambda\eta(|\,x\,|)\,\eta(x'\,|\,|\,x) \\
-\lambda\eta(x'\,|\,x\,|\,x) + i\,\delta(x - x') = 0,
\end{aligned}
\right\} \quad (4.23)
$$

$$
\left.
\begin{aligned}
(\partial_y + M)\,\eta(|\,y\,|\,z) + \lambda\eta(y\,|\,|)\,\eta(|\,y\,|\,z) - \lambda\eta(y\,|\,|\,z)\,\eta(|\,y\,|) \\
+\lambda\eta(y\,|\,y\,|\,z) + i\,\delta(y - z) = 0,
\end{aligned}
\right\} \quad (4.24)
$$

$$
\left.
\begin{aligned}
(\bar{\partial}_z + M)\,\eta(|\,y\,|\,z) + \lambda\eta(z\,|\,|)\,\eta(|\,y\,|\,z) + \lambda\eta(z\,|\,y\,|)\,\eta(|\,|\,z) \\
+\lambda\eta(z\,|\,y\,|\,z) + i\,\delta(y - z) = 0,
\end{aligned}
\right\} \quad (4.25)
$$

and, finally, an equation involving three space time points

$$\left.\begin{aligned}(\partial_y + M)\,\eta\,(x\,|\,y\,|\,z) + \lambda\eta\,(y\,|\,|\,)\,\eta\,(x\,|\,y\,|\,z) + \lambda\eta\,(x y\,|\,y\,|\,z)\\ + \lambda\eta\,(x y\,|\,|\,)\,\eta\,(\,|\,y\,|\,z) - \lambda\eta\,(x y\,|\,|\,z)\,\eta\,(\,|\,y\,|\,)\\ - \lambda\eta\,(y\,|\,|\,z)\,\eta\,(x\,|\,y\,|\,) = 0\,.\end{aligned}\right\}\quad(4.26)$$

The last equation can, for instance, be obtained by operating with $i\delta/\delta I\,(x)$ on the equation (4.24).

In the CSR it would seem most natural to represent the state under consideration by the state vector amplitudes taken in the limit of vanishing sources. There is, however, as emphasized by Schwinger [2], some advantage of postponing the limiting process $I\,(x) \to 0$ to a later stage in the considerations. If we, instead of considering meson theory, had taken electrodynamics as an example of illustrating the general scheme developed here, we would have had an obvious reason for doing this, as in electrodynamics the external source of the electromagnetic field has a direct interpretation in terms of a classical distribution of current and charge interacting with the system. Such a justification can hardly be found in our case. Still, we shall find it mathematically convenient in the following considerations to keep the meson field source in the theory.

We, thus, consider the limit of vanishing spinor sources. In this case, simplifications arise due to the fact that the difference ΔN between the total number of nucleons and the total number of anti-nucleons is then a constant of the motion. This implies a selection rule for T_0-functions. Only those T_0-functions are different from zero which contain the same number of nucleon and anti-nucleon space time points. If no $\boldsymbol{T}$ operator appeared in the definition

$$T_0\,(x'\cdots|\,y'\cdots|\,x'\cdots) = \langle 0\,|\,\boldsymbol{T}\cdot T\,(u\,(x')\cdots\psi\,(y')\cdots\overline{\psi}\,(z')\cdots)\,|\,0\rangle,$$

this selection rule would follow in the usual way from $\Delta N\,|\,0\rangle = 0$. However, it is easily seen from (2.5), remembering that in the limit considered we have $W = I\,u$, that ΔN commutes with $\boldsymbol{T}$ and, thus, the selection rule is not influenced by the presence of the $\boldsymbol{T}$-operator.

With this result, we can write (4.5) in the simpler form

$$\left.\begin{aligned}(-\Box_x + m^2)\,\Psi(x\,|\,|\,) - \lambda\,\Psi(\,|\,x\,|\,x) &= 0,\\ (\vartheta_y + M)\,\Psi(\,|\,y\,|\,) + \lambda\eta\,(y\,|\,|\,)\,\Psi(\,|\,y\,|\,) + \lambda\,\Psi(y\,|\,y\,|\,) &= 0,\\ (\bar{\vartheta}_z + M)\,\Psi(\,|\,|\,z) + \lambda\eta\,(z\,|\,|\,)\,\Psi(\,|\,|\,z) + \lambda\,\Psi(z\,|\,|\,z) &= 0,\end{aligned}\right\}\quad(4.27)$$

the limit $\varphi = \bar{\varphi} = 0$ being understood in these equations. It may be of some interest in the following to compare these equations with the equations obtained from (4.23), (4.24), and (4.25), taking the same limit, viz.

$$\left.\begin{aligned}(-\Box_x + m^2)\,\eta\,(xx'\,|\,|\,) - \lambda\eta\,(x'\,|\,x\,|\,x) + i\delta\,(x - x') &= 0,\\ (\vartheta_y + M)\,\eta\,(\,|\,y\,|\,z) + \lambda\eta\,(y\,|\,|\,)\,\eta\,(\,|\,y\,|\,z) + \lambda\eta\,(y\,|\,y\,|\,z) + i\delta\,(y - z) &= 0,\\ (\bar{\vartheta}_z + M)\,\eta\,(\,|\,y\,|\,z) + \lambda\eta\,(z\,|\,|\,)\,\eta\,(\,|\,y\,|\,z) + \lambda\eta\,(z\,|\,y\,|\,z) + i\delta\,(y - z) &= 0.\end{aligned}\right\}\quad(4.28)$$

These two sets of equations are of very much the same structure. The main difference is that the equations for the state vector amplitudes are homogeneous equations, while those for the η-functions are inhomogeneous ones. We shall discuss the relations between these two sets of equations more closely in the next section. Here, we only mention that the second equation (4.27) and the second equation (4.28) may be written as

$$\left.\begin{aligned}\left(\vartheta_y + M + \lambda\eta\,(y\,|\,|\,) + i\lambda\,\frac{\delta}{\delta I(y)}\right)\Psi\,(\,|\,y\,|\,) &= 0,\\ \left(\vartheta_y + M + \lambda\eta\,(y\,|\,|\,) + i\lambda\,\frac{\delta}{\delta I(y)}\right)\eta\,(\,|\,y\,|\,z) &= -\,i\delta\,(y - z),\end{aligned}\right\}\quad(4.29)$$

respectively. Thus, we see that, in a certain sense, $\Psi(\,|\,y\,|\,)$ obey the homogeneous equation of motion corresponding to the equation for $\eta\,(\,|\,y\,|\,z)$.

5. The equations for the one-and two-nucleon problems.

As mentioned in the Introduction, the present formalism combines the theory of Schwinger with that of Heisenberg and Freese. To illustrate this we shall briefly discuss the formal properties of the one- and two-nucleon equations from the point of view of the CSR. For the sake of completeness, and in order to introduce some convenient notations, we first summarize the derivation of the one-nucleon equation given by SCHWINGER [2].

i) *The one-nucleon equation.*

In the limit of vanishing spinor sources the simplest η-functions satisfy, according to (4.10) and (4.28), the equations of motion

$$\left.\begin{aligned} (-\Box + m^2)\,\eta\,(x\,|\,|) - \lambda\eta\,(|\,x\,|\,x) + I\,(x) &= 0, \\ (-\Box + m^2)\,\eta\,(xx'\,|\,|) - \lambda\eta\,(x'\,|\,x\,|\,x) + i\delta\,(x-x') &= 0, \\ (\partial_y + m + \lambda\eta\,(y\,|\,|))\,\eta\,(|\,y\,|\,z) + \lambda\eta\,(y\,|\,y\,|\,z) + i\delta\,(y-z) &= 0. \end{aligned}\right\} \quad (5.1)$$

To simplify the notation, and also to distinguish the η-functions in this limit from the general ones, we introduce

$$\left.\begin{aligned} U\,(x) &= \eta\,(x\,|\,|), \\ \Delta_c'\,(x, x') &= i\eta\,(xx'\,|\,|), \\ S_c'\,(x, x') &= -\,i\eta\,(|\,x\,|\,x'), \end{aligned}\right\} \quad (5.2)$$

and, consequently, write the equations (5.1) in the form

$$\left.\begin{aligned} (-\Box + m^2)\,U\,(x) - i\,\lambda\,S_c'\,(x, x) + I\,(x) &= 0, \\ (-\Box + m^2)\,\Delta_c'\,(x, x') + i\,\lambda\,\frac{\delta}{\delta I\,(x')}\,S_c'\,(x, x) &= \delta\,(x-x'), \\ (\partial + M + \lambda\,U\,(x))\,S_c'\,(x, x') + i\,\lambda\,\frac{\delta}{\delta I(x)}\,S_c'\,(x, x') &= -\,\delta\,(x-x'). \end{aligned}\right\} \quad (5.3)$$

As discussed in Section 4.ii, the η-functions satisfy causal boundary conditions. Hence, in the limit $I = \lambda = 0$, we have

$$\left.\begin{aligned} \Delta_c'\,(x, x') &= \Delta_c\,(x-x'), \\ S_c'\,(x, x') &= S_c\,(x-x'), \end{aligned}\right\} \quad (5.4)$$

where Δ_c and S_c are the well-known causal solutions of

$$\left.\begin{aligned} (-\Box + m^2)\,\Delta_c\,(x-x') &= \delta\,(x-x'), \\ (\partial + M)\,S_c\,(x-x') &= -\,\delta\,(x-x'). \end{aligned}\right\} \quad (5.5)$$

The equations (5.3) are, for our case of nucleons interacting with scalar neutral mesons, the analogue of the equations for the Green's functions in electrodynamics studied in Schwinger's paper. Following his method we substitute the variational derivative operators in (5.3) by polarization operators Π_c^* and Σ_c^* defined by

$$\left.\begin{aligned} i\lambda \frac{\delta}{\delta I(x')} S_c'(x,x) &= \Pi_c^*(x,1)\,\Delta_c'(1,x') \\ i\lambda \frac{\delta}{\delta I(x)} S_c'(x,x') &= \Sigma_c^*(x,1)\,S_c'(1,x'). \end{aligned}\right\} \tag{5.6}$$

Here, and in the following, numbers occurring twice denote variables of integration. Thus, for instance,

$$\Sigma_c^*(x,1)\,S_c'(1,x') = \int \Sigma_c^*(x,\xi')\,S_c'(\xi',x')\,d\xi'. \tag{5.7}$$

By (5.6) the equations (5.3) take the form

$$\left.\begin{aligned} (-\Box + m^2)\,U(x) - i\lambda S_c'(x,x) &= -I(x), \\ (-\Box + m^2)\,\Delta_c'(x,x') + \Pi_c^*(x,1)\,\Delta_c'(1,x') &= \delta(x-x'), \\ (\partial\!\!\!/ + M + \lambda U(x))\,S_c'(x,x') + \Sigma_c^*(x,1)\,S_c'(1,x') &= -\delta(x-x'). \end{aligned}\right\} \tag{5.8}$$

Operating on the last of these equations with $i\lambda\delta/\delta I(x'')$ we get, after integration and taking into account the causal boundary conditions,

$$i\lambda \frac{\delta}{\delta I(x'')} S_c'(x',x''') = S_c'(x',1)\,\Phi(1,2,3)\,\Delta_c'(2,x'')\,S_c'(3,x'''). \tag{5.9}$$

The kernel Φ depending on three space time points is given by

$$\left.\begin{aligned} &\Phi(x',x'',x''') \\ &= -i\lambda^2\,\delta(x'-x'')\,\delta(x''-x''') - i\lambda \frac{\delta\Sigma_c^*(x',x''')}{\delta U(x'')}. \end{aligned}\right\} \tag{5.10}$$

In the derivation of (5.9) use has been made of the fact that I does not appear explicitly in the last equation (5.8), whence

$$\frac{\delta}{\delta I(x)} \Sigma_c^* (x', x''') = - \frac{\delta \Sigma_c^* (x', x''')}{\delta U(2)} \Delta_c' (2, x''). \quad (5.11)$$

On comparison of (5.6) and (5.9) we infer integro variational equations characterizing the polarization operators, viz.

$$\left. \begin{aligned} \Pi_c^* (x, x') &= S_c' (x, 1)\, \Phi (1, x', 3)\, S_c' (3, x), \\ \Sigma_c^* (x, x') &= S_c' (x, 1)\, \Phi (1, 2, x')\, \Delta_c' (2, x). \end{aligned} \right\} \quad (5.12)$$

For later reference we mention that, from the equation conjugate to the last equation (5.1), viz.

$$\left(\bar{\partial}_z + M + \lambda \eta (z\,|\,|)\right) \eta (|\,y\,|\,z) + \lambda \eta (z\,|\,y\,|\,z) + i\, \delta (y - z) = 0, \quad (5.13)$$

we get by arguments similar to those leading to (5.8) an equation of the form

$$\left(\bar{\partial}' + M + \lambda\, U (x')\right) S_c' (x, x') + S_c' (x, 1)\, \bar{\Sigma}_c^* (1, x') = - \delta (x - x'). \quad (5.14)$$

The polarization operator in this equation is given by

$$\bar{\Sigma}_c^* (x, x') = \Phi (x, 2, 3)\, \Delta_c' (2, x')\, S_c' (3, x'). \quad (5.15)$$

According to Schwinger, the one-nucleon equation is obtained as the homogeneous equation of motion corresponding to the inhomogeneous equation (5.8) for the Green's function S_c'. Thus, denoting the one-nucleon "wave function" by χ, the equation reads

$$\left(\partial + M + \lambda\, U (x)\right) \chi (x) + \Sigma_c^* (x, 1)\, \chi (1) = 0. \quad (5.16)$$

As shown in the previous section, the equation of motion for the state vector amplitude depending on one nucleon coordinate is

$$\left(\partial_y + M + \lambda\, U (y)\right) \Psi (|\,y\,|) + i \lambda \frac{\delta}{\delta I (y)} \Psi (|\,y\,|) = 0. \quad (5.17)$$

The similarity between this equation and the inhomogeneous equation for the Green's function S_c' makes it natural to investigate under which conditions solutions of (5.16) also satisfy (5.17). For this to be true we must have

$$i\lambda \frac{\delta}{\delta I(x)} \chi(x) = \Sigma_c^*(x,1)\,\chi(1). \tag{5.18}$$

By derivation of (5.16) with respect to $I(x')$ we get, after integration, an expression for the variational derivative of χ, viz.

$$i\lambda \frac{\delta \chi(x)}{\delta I(x')} = \chi^{(0)}(x';x) + S_c'(x,1)\,\Phi(1,2,3)\,\Delta_c'(2,x')\,\chi(3). \tag{5.19}$$

The function $\chi^{(0)}$ is a so far undetermined solution of

$$(\partial + M + \lambda\,U(x))\,\chi^{(0)}(x';x) + \Sigma_c^*(x,1)\,\chi^{(0)}(x';1) = 0. \tag{5.20}$$

Comparing (5.18) with (5.19) we see that χ is a solution of (5.17), provided that $\chi^{(0)}$ vanishes.

We thus have the result that any solution of the coupled equations

$$\left.\begin{aligned} &(\partial_y + M + \lambda\,U(y))\,\Psi_{(1)}(|y|) + \Sigma_c^*(y,1)\,\Psi_{(1)}(|1|) = 0,\\ &\Psi_{(1)}(x|y|) = \lambda^{-1}\,S_c'(y,1)\,\Phi(1,2,3)\,\Delta_c'(2,x)\,\Psi_{(1)}(|3|) \end{aligned}\right\} \tag{5.21}$$

also satisfies (5.17). The reverse statement is of course not true. We have, therefore, attached a subscript to the state vector amplitudes in these equations to indicate that a solution in the form (5.21) is possible for a restricted class of states only, the one-nucleon states.

From (5.17) we get by variational derivation an infinite system of coupled equations for the state vector amplitudes. The first of the equations derived from (5.17) reads

$$\left.\begin{aligned} (\partial_y + M + \lambda\,U(y))\,\Psi(x|y|) - i\lambda\,\Delta_c'(x,y)\,\Psi(|y|)&\\ + \lambda\,\Psi(xy|y|) = 0.& \end{aligned}\right\} \tag{5.22}$$

Let us now follow, in the present version of the CSR, the suggestion by Freese and try to eliminate all amplitudes depending on one or more meson coordinates from the infinite set of equations. The states for which this elimination process is possible might, alternatively, be called the one-nucleon states. To get an idea how the resulting equation will look we convert the infinite system of equations into a finite one by the approximation assumption that $\Psi(xy|y|)$ can be neglected in (5.22). We can then solve (5.22) by the aid of the Green's function satisfying

$$(\partial_y + M + \lambda\, U(y))\, S_c^{(U)}(y, y') = -\,\delta(y - y'), \qquad (5.23)$$

and causal boundary conditions. The approximate solution of (5.22) is then

$$\Psi(x\,|\,y\,|) = \varphi^{(0)}(x;\, y) - i\,\lambda\, S_c^{(U)}(y, 1)\, \Delta_c'(1, x)\, \Psi(|\,1\,|), \qquad (5.24)$$

where $\varphi^{(0)}$ is a solution of

$$(\partial_y + M + \lambda\, U(y))\, \varphi^{(0)}(x;\, y) = 0. \qquad (5.25)$$

To obtain an equation of the form (5.16) we choose $\varphi^{(0)} = 0$. With this choice we get, instead of (5.17),

$$(\partial_y + M + \lambda\, U(y))\, \Psi(|\,y\,|) - i\,\lambda^2 S_c^{(U)}(y, 1)\, \Delta_c'(1, y)\, \Psi(|\,1\,|) = 0. \qquad (5.26)$$

In the next approximation one would keep all amplitudes with less than two meson space time points. Proceeding in this way one can, in principle, construct an exact equation of the form (5.16), provided that the procedure converges. The polarization operator $'\Sigma_c^*$, say, obtained in this way is characterized by the requirement that the resulting equation

$$(\partial_y + M + \lambda\, U(y))\, \Psi(|\,y\,|) + {'\Sigma_c^*}(y, 1)\, \Psi(|\,1\,|) = 0$$

is consistent with (5.17), i. e. that

$${'\Sigma_c^*}(y, 1)\, \Psi(|\,1\,|) = i\,\lambda\, \frac{\delta}{\delta I(y)}\, \Psi(|\,y\,|).$$

By arguments similar to those above it can easily by verified that $'\Sigma_c^*$ is, in fact, identical with Σ_c^*. Thus the resulting one-nucleon equation is identical with Schwinger's equation.

The advantage of the equations (5.21) as compared with the infinite system of equations obtained from (5.17) becomes obvious when we pass to the physically interesting limit of vanishing external sources. For $I = 0$, the second equation (5.21) and the equations obtained therefrom by variational derivation become explicit expressions for the state vector amplitudes with one and more meson space time coordinates. Therefore, for $I = 0$, any solution of the one-nucleon equation provides us with a corresponding solution of the equations of motion in the configuration space representation.

ii) *The two-nucleon equation.*

According to Schwinger, the two-nucleon Green's function is defined by

$$G(y, y'; z, z') = \frac{\delta}{\delta f(z')} \frac{\delta}{\delta f(z)} \frac{T_0(|yy'|)}{T_0(||)}\bigg|_{\varphi=\bar{\varphi}=0.} \tag{5.27}$$

Using the formulas in Section 4, it is easily verified that

$$\left.\begin{aligned} & G(y, y'; z, z') \\ & = \eta(|yy'|zz') - \eta(|y|z)\,\eta(|y'|z') + \eta(|y|z')\,\eta(|y'|z), \end{aligned}\right\} \tag{5.28}$$

the limit $\varphi = \bar{\varphi} = 0$ being understood in this formula. An equation of motion for $\eta(|yy'|zz')$ can be obtained from (4.10) by taking appropriate variational derivatives. From the equation obtained in this way, and by (5.3), we get

$$\left.\begin{aligned} & \left(\partial_y + M + \lambda U(y)\right) G(y, y'; z, z') + i\lambda \frac{\delta}{\delta I(y)} G(y, y'; z, z') \\ & \quad = -\delta(y-z)\, S_c'(y', z') + \delta(y-z')\, S_c'(y', z). \end{aligned}\right\} \tag{5.29}$$

Using (5.3) we see that

$$\left.\begin{aligned} & \left(\partial_{y'} + M + \lambda U(y) + \lambda \frac{\delta}{\delta I(y)}\right)\left(\partial_y + M + \lambda U(y') + i\lambda \frac{\delta}{\delta I(y')}\right) G(y, y'; z, z') \\ & \quad = \delta(y-z)\,\delta(y'-z') - \delta(y-z')\,\delta(y'-z). \end{aligned}\right\} \tag{5.30}$$

Following Schwinger, we introduce an interaction operator W by

$$\left.\begin{aligned} & F(y)\, F(y')\, G(y, y'; z, z') - W(y, y'; 1, 2)\, G(1, 2; z, z') \\ & \quad = \delta(y-z)\,\delta(y'-z') - \delta(y-z')\,\delta(y'-z). \end{aligned}\right\} \tag{5.31}$$

The symbol F is an abbreviation of the integral differential operator entering in the equation for the one-nucleon Green's function, i. e.

$$F(y)\,\xi(y) = \left(\partial_y + M + \lambda U(y)\right)\xi(y) + \Sigma_c^*(y, 1)\,\xi(1). \tag{5.32}$$

As G satisfies causal boundary conditions we can integrate (5.31) by means of S_c'. Combining the resulting equation, viz.

$$\left.\begin{aligned} &F(y)\,G(y,y';z,z') + S_c'(y',2)\,W(y,2;3,4)\,G(3,4;z,z') \\ &\quad = -\delta(y-z)\,S_c'(y',z') + \delta(y-z')\,S_c'(y',z), \end{aligned}\right\} \quad (5.33)$$

with (5.29) we infer a condition on the interaction operator:

$$\left.\begin{aligned} &i\lambda\frac{\delta}{\delta I(y)}\,G(y,y';z,z') = \Sigma_c^*(y,1)\,G(1,y';z,z') \\ &\quad + S_c'(y',2)\,W(y,2;3,4)\,G(3,4;z,z'). \end{aligned}\right\} \quad (5.34)$$

Integrating (5.33) once more, we find that

$$\left.\begin{aligned} &G(y,y';z,z') - S_c'(y,1)\,S_c'(y',2)\,W(1,2;3,4)\,G(3,4;z,z') \\ &\quad = S_c'(y,z)\,S_c'(y',z') - S_c'(y,z')\,S_c'(y',z). \end{aligned}\right\} \quad (5.35)$$

From this equation one gets an expression for the variational derivative of G with respect to $I(y)$ which, together with (5.34), gives Schwinger's characterization of the interaction operator, viz.

$$\left.\begin{aligned} &W(y,y';1,2)\,G(1,2;z,z') \\ &\quad = \Phi(y',1,2)\,\Delta_c'(1,y)\,G(y,2;z,z') \\ &\quad + S_c'(y,1)\,i\lambda\frac{\delta}{\delta I(y)}\,[W(1,y';3,4)\,G(3,4;z,z')]. \end{aligned}\right\} \quad (5.36)$$

For W we shall use another equation which does not depend explicitly on the variational derivative of the two-nucleon Green's function. From (5.35) one gets, using (5.9),

$$\left.\begin{aligned} &i\lambda\frac{\delta}{\delta I(x)}\,G(y,y';z,z') \\ &= \frac{1}{2}\lambda\,G(y,y';1,2)\left[i\frac{\delta}{\delta I(x)}\,W(1,2;3,4)\right]G(3,4;z,z') \\ &+ G(y,y';1,2)\,\Phi(1,3,4)\,\Delta_c'(3,x)\,S_c'(4,5)\,F(5)\,F(2)\,G(5,2;z,z'). \end{aligned}\right\} \quad (5.37)$$

The combination of this expression with (5.34) gives the alternative characterization of W, viz.

$$\left.\begin{aligned}\frac{1}{2}\left(\Sigma_c^*(y,z)\,\delta(y'-z')-\Sigma_c^*(y,z')\,\delta(y'-z)\right)+S_c'(y',2)\,W(y,2;z,z')\\ =\frac{1}{2}\lambda\,G(y,y';1,2)\,i\frac{\delta}{\delta I(y)}W(1,2;z,z')\\ -\frac{1}{2}\overline{F}(z')\,G(y,y';1,z')\,\Phi(1,2,z)\,\Delta_c'(2,y)\\ -\frac{1}{2}\overline{F}(z)\,G(y,y';z,1)\,\Phi(1,2,z')\,\Delta_c'(2,y).\end{aligned}\right\}\quad(5.38)$$

Here, $\overline{F}$ denotes the operator entering in the equation of motion for the one-nucleon Green's function in the form given by (5.14), i. e.

$$\overline{F}(z)\,\zeta(z)=\left(\overline{\partial}_z+M+\lambda\,U(z)\right)\zeta(z)+\zeta(1)\,\overline{\Sigma}_c^*(1,z).\quad(5.39)$$

The equations of motion for the state vector amplitude depending on two nucleon space time points obtained from the equation (4.22) by passing to the limit of vanishing spinor sources read

$$\left.\begin{aligned}\left(\partial_y+M+\lambda\,U(y)\right)\Psi(|yy'|)+\lambda\,\Psi(y|yy'|)=0,\\ \left(\partial_{y'}+M+\lambda\,U(y')\right)\Psi(|yy'|)+\lambda\,\Psi(y'|yy'|)=0,\end{aligned}\right\}\quad(5.40)$$

whence also

$$\left(\partial_y+M+\lambda\,U(y)+i\lambda\frac{\delta}{\delta I(y)}\right)\left(\partial_{y'}+M+\lambda\,U(y')+i\lambda\frac{\delta}{\delta I(y')}\right)\Psi(|yy'|)=0.\quad(5.41)$$

This equation is a homogeneous equation of motion corresponding to (5.30) in the same sense as the equation (5.17) for the one-nucleon amplitude is the homogeneous equation corresponding to the equation for the one-nucleon Green's function (5.3).

According to Schwinger, the two-nucleon equation is the homogeneous equation corresponding to the equation (5.31), i. e.

$$F(y)\,F(y')\,\chi(y,y')-W(y,y';1,2)\,\chi(1,2)=0.\quad(5.42)$$

It seems to be difficult to establish any general connection between the solutions of this equation and the solutions of (5.41). If, however, we take instead of (5.42) the two integrated equations

$$\left.\begin{array}{l} F(y)\,\chi(y,y') + S_c'(y',2)\,W(y,2;3,4)\,\chi(3,4) = 0, \\ F(y')\,\chi(y,y') + S_c'(y,1)\,W(1,y';3,4)\,\chi(3,4) = 0, \end{array}\right\} \quad (5.43)$$

where the inhomogeneous terms have been dropped, then one can rather easily find the connection between the solutions of these equations and the solutions of (5.40). Evidently, the condition for compatibility of (5.43) and (5.40) is that

$$\left.\begin{array}{l} \Sigma_c^*(y,1)\,\chi(1,y') + S_c'(y',2)\,W(y,2;3,4)\,\chi(3,4) \\ \qquad = i\lambda\,\dfrac{\delta}{\delta I(y)}\,\chi(y,y'), \\ \Sigma_c^*(y',2)\,\chi(y,2) + S_c'(y,1)\,W(1,y';3,4)\,\chi(3,4) \\ \qquad = i\lambda\,\dfrac{\delta}{\delta I(y')}\,\chi(y,y'). \end{array}\right\} \quad (5.44)$$

By integration of (5.43) we get

$$\chi(y,y') - S_c'(y,1)\,S_c'(y',2)\,W(1,2;3,4)\,\chi(3,4) = \varphi(y,y'), \quad (5.45)$$

where φ is any solution of

$$F(y)\,\varphi(y,y') = F(y')\,\varphi(y,y') = 0, \quad (5.46)$$

i. e. φ has one-particle properties with respect to both coordinates. From this equation we infer by arguments similar to those used in the derivation of the one-particle equation that

$$\left.\begin{array}{l} i\lambda\,\dfrac{\delta}{\delta I(x)}\,\varphi(y,y') = \varphi^{(0)}(x;y,y') \\ \qquad + S_c'(y',1)\,\Phi(1,2,3)\,\Delta_c'(2,x)\,\varphi(y,3) \\ \qquad + S_c'(y,1)\,\Phi(1,2,3)\,\Delta_c'(2,x)\,\varphi(3,y'), \end{array}\right\} \quad (5.47)$$

where

$$F(y)\,\varphi^{(0)}(x;y,y') = F(y')\,\varphi^{(0)}(x;y,y') = 0. \quad (5.48)$$

Using this we find from (5.45) an expression for the variational derivative of χ with respect to $I(x)$, viz.

$$\left.\begin{array}{l} i\lambda\,\dfrac{\delta}{\delta I(x)}\,\chi(y,y') = \dfrac{1}{2}\,\lambda\,G(y,y';1,2)\left[i\,\dfrac{\delta}{\delta I(x)}\,W(1,2;3,4)\right]\chi(3,4) \\ \quad - \left(\bar{F}(4)\,G(y,y';1,4)\right)\Phi(1,2,3)\,\Delta_c'(2,x)\,\chi(3,4) + R^{(0)}, \end{array}\right\} \quad (5.49)$$

where $R^{(0)}$ is a contribution which vanishes for $\varphi^{(0)}$ equal to zero. On comparison with (5.44) and using (5.38) we find that χ satisfies (5.40), provided that $\varphi^{(0)}$ vanishes.

Hence, corresponding to (5.21), we have the result for the two-nucleon system: Any solution of the coupled equations

$$\left.\begin{aligned} &\Psi_{(2)}(|yy'|) - S_c'(y,1)\,S_c'(y',2)\,W(1,2;3,4)\,\Psi_{(2)}(|34|) = \varphi(y,y') \\ &i\lambda\frac{\delta}{\delta I(x)}\Psi_{(2)}(|yy'|) = \frac{1}{2}\lambda G(y,y';1,2)\left[i\frac{\delta}{\delta I(x)}W(1,2;3,4)\right]\Psi_{(2)}(|34|) \\ &\quad - \left(\bar{F}(4)\,G(y,y';1,4)\right)\Phi(1,2,3)\,\Delta_c'(2,x)\,\Psi_{(2)}(|34|), \end{aligned}\right\} \quad (5.50)$$

where $\varphi(y, y')$ satisfies (5.46), is a solution of (5.40). In particular, passing to the limit $I = 0$, the second equation (5.50) and its variational derivatives become explicit expressions for the state vector amplitudes depending on one and more meson coordinates besides the two nucleon space time coordinates. It is thus possible in a unique way to relate to any solution of the Bethe-Salpeter equation a solution of the equations of motion in the configuration space representation.

Summary.

A reformulation of quantum field theory is given, in which any state of the system considered is represented by a functional depending on external sources. The variational derivatives of this functional provide us with a generalization of the Fock representation in configuration space to the case of non-linear fields. The representing amplitudes can be expressed entirely in terms of matrix elements of time ordered products of field operators and possess several simple properties which are independent of the magnitude of the coupling constant. It is shown that these amplitudes satisfy homogeneous equations of motion which can be derived in a simple manner. The equations of the Bethe-Salpeter type following herefrom become identical with those following from Schwinger's theory of Green's functions. Our representation has many properties in common with that given by Freese.

Appendix I.

The sources of the spinor fields.

In Section 1 we have assumed that the domain of the external sources can be chosen so large that variational derivatives with respect to allowed variations of f and g can be defined in a unique way. This property together with the anti-commutativity (1.6) is all we need for the development of the configuration space representation. It is, maybe, not quite trivial that the requirements to the sources are consistent. We shall, therefore, construct an example of a possible domain of allowed f-number pairs.

Let a_n and b_n, $n = 1, 2, \cdots$ be two sets of infinite matrices which satisfy the commutation rules

$$\left.\begin{aligned} \{a_n, a_m^\dagger\} &= \delta_{nm} \\ \{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} &= 0, \\ \{b_n, b_m^\dagger\} &= \delta_{nm} \\ \{b_n, b_m\} = \{b_n^\dagger, b_m^\dagger\} &= 0, \end{aligned}\right\} \quad \text{(Ap. I. 1)}$$

while all the a's anti-commute with all the b's. As is well known, there exists a matrix which anti-commutes with all the a's and with all the b's and with their adjoints. This matrix Ω, say, is the parity of the matrix $\Sigma\,(a_n^\dagger\, a_n + b_n^\dagger\, b_n)$. We choose Ω hermitian and unitary, i. e.

$$\Omega^\dagger = \Omega, \quad \Omega^2 = 1. \qquad \text{(Ap. I. 2)}$$

For the construction of the f-number pairs we further need two complete orthonormal sets of functions in four-dimensional space, $f_n(x)$ and $g_n(x)$, such that any function, $\xi(x)$ say, can be expanded in either of the forms

$$\xi(x) = \Sigma\, \xi_n^{(f)}\, f_n(x)$$

or

$$\xi(x) = \Sigma\, \xi_n^{(g)}\, g_n(x).$$

Let c_1 and c_2 be complex numbers. Then,

$$\left.\begin{aligned} f(x) &= c_1 \Sigma\, a_n f_n(x) \\ g(x) &= c_2 \Sigma\, b_n g_n(x), \end{aligned}\right\} \qquad \text{(Ap. I. 3)}$$

is a possible allowed f-number pair. In fact, due to (Ap. I. 1)

$$\{f(x), f(x')\} = \{f(x), g(x')\} = \{g(x), g(x')\} = 0. \qquad \text{(Ap. I. 4)}$$

A domain of f-number pairs can be obtained from the particular pair (Ap. I. 3) by unitary transformations in the a, b-space. In particular we are interested in infinitesimal unitary transformations such that the corresponding variations of the f-number pair form a pair of allowed variations in the sense of Section 1, i. e. such that

$$\left.\begin{aligned} \{\delta f(x), f(x')\} &= \{\delta f(x), g(x')\} = 0, \\ \{\delta g(x), f(x')\} &= \{\delta g(x), g(x')\} = 0. \end{aligned}\right\} \qquad \text{(Ap. I. 5)}$$

Such variations can be obtained by means of the matrix

$$A = \Sigma\, (a_n^\dagger\, \Omega\, \alpha_n - \alpha_n^*\, \Omega\, a_n + b_n^\dagger\, \Omega\, \beta_n - \beta_n^*\, \Omega\, b_n), \qquad \text{(Ap. I. 6)}$$

where the α_n's and the β_n's are infinitesimal complex numbers. By the properties of Ω, A is anti-hermitian, whence $1 + A$ is unitary. By this transformation the a_n's and the b_n's vary according to

$$\left.\begin{aligned} \delta a_n &= -[A, a_n] = \Omega \alpha_n, \\ \delta b_n &= -[A, b_n] = \Omega \beta_n. \end{aligned}\right\} \qquad \text{(Ap. I. 7)}$$

The corresponding variation of the f-number pairs is

$$\left.\begin{aligned} \delta f(x) &= c_1\, \Omega\, \Sigma\, \alpha_n f_n(x), \\ \delta g(x) &= c_2\, \Omega\, \Sigma\, \beta_n g_n(x). \end{aligned}\right\} \qquad \text{(Ap. I. 8)}$$

Obviously we have here an example of a pair of allowed variations for any set of infinitesimal α_n's and β_n's. Thus, all variations of the form

$$\left.\begin{aligned} \delta f(x) &= \Omega\, \delta\xi(x), \\ \delta g(x) &= \Omega\, \delta\eta(x), \end{aligned}\right\} \qquad \text{(Ap. I. 9)}$$

where $\delta\xi$ and $\delta\eta$ are infinitesimal functions are included among the allowed variations. Therefore, if an expression of the form (1.10) holds for an arbitrary pair of allowed variations, we have in particular

$$\Omega \int \left(\delta\xi\,(x)\,K\,(x) + \delta\eta\,(x)\,L\,(x)\right)\delta x = 0\,, \qquad \text{(Ap. I. 10)}$$

with arbitrary $\delta\,\xi$ and $\delta\,\eta$. As Ω is non-singular we conclude that $K\,(x)$ as well as $L\,(x)$ vanish identically.

Appendix II.

Reformulation of a theorem due to Wick.

Let $u\,(x)$ be the field operator of a free scalar neutral meson field. We shall use Dyson's notation

$$N\left(u\,(x')\,u\,(x'')\cdots u\,(x^{(n)})\right) \qquad \text{(Ap. II. 1)}$$

to designate the product of the u's ordered such that all absorption operators stand to the right of all emission operators. This product we call the normal product of the u's indicated. As shown by Wick [9], any time ordered product can be decomposed into a sum of normal constituents according to

$$T\left(u\,(x')\,u\,(x'')\cdots u\,(x^{(n)})\right) = \sum_{\nu=0}^{n} N^{(\nu)}. \qquad \text{(Ap. II. 2)}$$

For ν odd $N^{(\nu)}$ vanishes. For ν even, $N^{(\nu)}$ is a sum of terms, one term for each possible pairing of ν factors u. Let for ν even, $\xi', \xi'', \cdots, \xi^{(\nu)}$ be some of the space time points $x', x'', \cdots, x^{(n)}$. For a definite pairing $(\xi', \xi''), (\xi''', \xi''''), \cdots (\xi^{(\nu-1)}, \xi^{(\nu)})$ the contribution to $N^{(\nu)}$ is

$$\left.\begin{aligned} &\langle 0\,|\,T\left(u(\xi')\,u(\xi'')\right)|\,0\rangle\langle 0\,|\,T\left(u(\xi''')\,u(\xi'''')\right)|\,0\rangle\cdots \\ &\qquad\times\langle 0\,|\,T\left(u\,(\xi^{(\nu-1)})\,u\,(\xi^{(\nu)})\right)|\,0\rangle \\ &\times N\left(u\,(x')\,u\,(x'')\cdots u\,(x^{(n)});\,\xi',\xi'',\cdots\xi^{(\nu)}\right). \end{aligned}\right\} \qquad \text{(Ap. II. 3)}$$

Here, $N\,(\cdots)$ denotes the normal product of the unpaired u's. For instance,

$$N\left(u\,(x')\,u\,(x'')\,u\,(x''')\,u(x'''');\,x''\,x''''\right) \equiv N\left(u\,(x')\,u\,(x''')\right).$$

To obtain $N^{(\nu)}$ from terms of the form (Ap. II. 3) one must add all contributions from possible pairings of the space time points $\xi', \xi'', \cdots \xi^{(\nu)}$ and, further, sum over all subsets of ν field operators u. Hence, we can write $N^{(\nu)}$ in the form

$$\left.\begin{array}{c} N^{(\nu)} = \sum\limits_{\xi', \xi'', \cdots \xi^{(\nu)}} C(\xi', \xi'', \cdots \xi^{(\nu)}) \\ N(u(x')\,u(x'')\cdots u(x^{(n)});\, \xi', \xi'', \cdots \xi^{(\nu)}), \end{array}\right\} \quad \text{(Ap. II. 4)}$$

where the C's are certain c-number functions not depending on n. The summation runs over all subsets $\xi', \xi'', \cdots \xi^{(\nu)}$. In particular, we note that, for n even,

$$N^{(n)} = C(x', x'', \cdots x^{(n)}). \qquad \text{(Ap. II. 5)}$$

Combining (Ap. II. 4) and (Ap. II, 2) we have

$$\left.\begin{array}{c} T\big(u(x')\cdots u(x^{(n)})\big) = \\ \sum\limits_{\nu=0}^{n} \sum\limits_{\xi'\cdots\xi^{(\nu)}} C(\xi', \cdots \xi^{(\nu)})\, N\big(u(x')\cdots u(x^{(n)});\, \xi', \cdots \xi^{(\nu)}\big). \end{array}\right\} \text{(Ap. II. 6)}$$

We include formally odd ν's in the summation and choose vanishing corresponding C's.

The vacuum expectation value of any N-product is zero. Thus, from (Ap. II. 6) for n even, we get explicit expressions for the C's, viz.

$$C(x', x'', \cdots x^{(n)}) = T_0(x', x'', \cdots x^{(n)}). \qquad \text{(Ap. II. 7)}$$

As in (2.20), $T_0(x', \cdots)$ stands for the vacuum expectation value of the T-product. Wick's theorem now takes the form

$$\left.\begin{array}{c} T\big(u(x')\cdots u(x^{(n)})\big) = \\ \sum\limits_{\nu=0}^{n} \sum\limits_{\xi'\cdots\xi^{(\nu)}} T_0(\xi', \cdots \xi^{(\nu)})\, N\big(u(x')\cdots u(x^{(n)});\, \xi', \cdots \xi^{(\nu)}\big). \end{array}\right\} \text{(Ap. II. 8)}$$

It should be noted that (Ap. II. 7) also holds for ν odd as the vacuum expectation value of the product of an odd number of free field operators vanishes.

In case also other types of fields are considered, the definition of the N-product is slightly modified. Each term in the N-product should now be multiplied by a factor ($\pm$) which has the value $+1$ if the permutation of spinor operators involved in the ordering process is even, and -1 if this permutation is odd.

Consider the case of a free meson field u and a free spinor field described by the field operators $\overline{\psi}$ and ψ. Similar to (Ap. II. 8) one can write Wick's theorem for this case in the form

$$\left.\begin{array}{l} T\left(u(x')\cdots u(x^{(k)})\,\psi(y')\cdots\psi(y^{(l)})\,\overline{\psi}(z')\cdots\overline{\psi}(z^{(m)})\right) = \\ \sum\limits_{\varkappa\lambda\mu}\sum\limits_{\xi'\cdots\xi^{(\varkappa)}}\sum\limits_{\eta'\cdots\eta^{(\lambda)}}\sum\limits_{\zeta'\cdots\zeta^{(\mu)}}(\pm)\,T_0(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}) \\ \times N(u(x')\cdots u(x^{(k)});\,\xi'\cdots\xi^{(\varkappa)}\,|\,\psi(y')\cdots\psi(y^{(l)});\,\eta'\cdots\eta^{(\lambda)}\,| \\ \qquad\overline{\psi}(z')\cdots\overline{\psi}(z^{(m)});\,\zeta'\cdots\zeta^{(\mu)}), \end{array}\right\}\quad\text{(Ap. II. 9)}$$

where ($\pm$) is the parity of the permutation

$$\left\{\begin{array}{c} (\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}) \\ (x'\cdots x^{(k)};\xi'\cdots\xi^{(\varkappa)}\,|\,y'\cdots y^{(l)};\,\eta'\cdots\eta^{(\lambda)}\,|\,z'\cdots z^{(m)};\,\zeta'\cdots\zeta^{(\mu)}) \\ \rightarrow(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}). \end{array}\right\}$$

We introduce the notation

$$\Psi(x'\cdots|\,y'\cdots|\,z'\cdots) = \langle 0\,|\,N(u(x')\cdots\psi(y')\cdots\overline{\psi}(z')\cdots)\,|\Psi\rangle. \quad\text{(Ap. II. 10)}$$

If, further, we use the notation of Section 2 (p. 17), we get from (Ap. II. 9)

$$\left.\begin{array}{l} T_{\Psi}(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}) = \\ \sum\limits_{\varkappa\lambda\mu}\frac{1}{\varkappa!}\sum\limits_{\xi'\cdots\xi^{(\varkappa)}}\frac{1}{\lambda!}\sum\limits_{\eta'\cdots\eta^{(\lambda)}}\frac{1}{\mu!}\sum\limits_{\zeta'\cdots\zeta^{(\mu)}}(\pm)\,T_0(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}) \\ \times\,\Psi(x'\cdots x^{(k)};\,\xi'\cdots\xi^{(\varkappa)}\,|\,y'\cdots y^{(l)};\,\eta'\cdots\eta^{(\lambda)}\,|\,z'\cdots z^{(m)};\,\zeta'\cdots\zeta^{(\mu)}). \end{array}\right\}\quad\text{(Ap. II. 11)}$$

The factorials take into account that we now perform the summation such that the ξ's, η's, and ζ's run independently over the x's, y's, and z's, respectively.

As is well known, the functions Ψ are the representing amplitudes for the state $|\Psi\rangle$ in the Fock representation in the configuration space. We can thus regard (Ap. II. 11) as a recursion formula expressing the Fock amplitudes in terms of matrix elements of time ordered products.

Appendix III.

The equations of motion for the T_{ψ}-functions.

The equations of motion for the T_{ψ}-functions depending on one space time point only are easily obtained from the field equations (1.2). One finds[1]

$$\left.\begin{aligned} &(-\Box_x + m^2)\, T_{\psi}(x|\,|) - \lambda\, T_{\psi}(|\,x\,|\,x) + I(x)\, T_{\psi}(|\,|) = 0,\\ &(\partial_y + M)\, T_{\psi}(|\,y\,|) + \lambda\, T_{\psi}(y\,|\,y\,|) + f(y)\, T_{\psi}(|\,|) = 0,\\ &(\bar{\partial}_z + M)\, T_{\psi}(|\,|\,z) + \lambda\, T_{\psi}(z\,|\,|\,z) + g\,(z)\, T_{\psi}(|\,|) = 0. \end{aligned}\right\}\text{(Ap. III. 1)}$$

From the variational equations (1.14) and the canonical commutators it follows that

$$(\partial_y + M)\,\delta\,\psi\,(y) = \delta\,[(\partial_y + M)\,\psi\,(y)],$$

and similar relations for the other field variables. Hence, the differential operators appearing in the field equations commute with all variational derivative operators. We can thus obtain equations of motion for T_{ψ}-functions depending on more than one space time point simply by taking variational derivatives of the equations (Ap. III. 1). For instance, applying $-\,i\,\delta/\delta\,f\,(z)$ to the second of these equations, we get

$$\left.\begin{aligned} &(\partial_y + M)\, T_{\psi}(|\,y\,|\,z) + \lambda\, T_{\psi}(y\,|\,y\,|\,z)\\ &+ f\,(y)\, T_{\psi}(|\,|\,z) + i\,\delta\,(y - z)\, T_{\psi}(|\,|) = 0. \end{aligned}\right\}\text{(Ap. III. 2)}$$

One should note that, for instance,

$$-\,i\,\frac{\delta}{\delta f(z)}\, T_{\psi}(|\,y\,|) = -\,T_{\psi}(|\,y\,|\,z).$$

[1] The T-product of $\psi\,(x)$ and $\bar{\psi}\,(x')$ for $x = x'$ is chosen as

$$T\,(\psi\,(x)\,\bar{\psi}\,(x)) = \frac{1}{2}\,[\psi\,(x),\,\bar{\psi}\,(x)].$$

Hence the minus sign in the first equation (Ap. III. 1).

The following equations hold

$$\left.\begin{aligned}&(-\Box_{x'}+m^2)\,T_{\Psi}(x'\cdots|y'\cdots y^{(l)}|z'\cdots)\\&\quad-(-1)^l\lambda\,T_{\Psi}(x''\cdots|x'y'\cdots y^{(l)}|x'z'\cdots)\\&\quad+I\,x')\,T_{\Psi}(x''\cdots|y'\cdots y^{(l)}|z'\cdots)\\&+i\sum_{\xi}\delta(x'-\xi)\,T_{\Psi}(x''\cdots;\xi|y'\cdots y^{(l)}|z'\cdots)=0.\end{aligned}\right\}\quad\text{(Ap. III. 3)}$$

$$\left.\begin{aligned}&(\partial_{y'}+M)\,T_{\Psi}(x'\cdots|y'\cdots|z'\cdots)\\&\quad+\lambda\,T_{\Psi}(x'\cdots y'|y'\cdots|z'\cdots)\\&\quad+f(y')\,T_{\Psi}(x'\cdots|y''\cdots|z'\cdots)\\&+i\sum_{\zeta}(\pm)\,\delta(y'-\zeta)\,T_{\Psi}(x'\cdots|y''\cdots|z'\cdots;\zeta)=0.\end{aligned}\right\}\quad\text{(Ap. III. 4)}$$

$$\left.\begin{aligned}&(\bar{\partial}_{z'}+M)\,T_{\Psi}(x'\cdots|y'\cdots y^{(l)}|z'\cdots)\\&\quad+\lambda\,T_{\Psi}(x'\cdots z'|y'\cdots y^{(l)}|z'\cdots)\\&\quad+g(z')(-1)^l\,T_{\Psi}(x'\cdots|y'\cdots y^{(l)}|z''\cdots)\\&+i\sum_{\eta}(\pm)\,\delta(\eta-z')\,T_{\Psi}(x'\cdots|y'\cdots y^{(l)};\eta|z''\cdots)=0.\end{aligned}\right\}\quad\text{(Ap. III. 5)}$$

The $(\pm)$ factors have the same meaning as, f. inst., in (2.25).

One can verify these formulas by induction on the number of space time points. To illustrate: if we apply $i\,\delta/\delta\,g(y)$ on (Ap. III.5), we get

$$\left.\begin{aligned}&(\bar{\partial}_{z'}+M)\,T_{\Psi}(x'\cdots|yy'\cdots y^{(l)}|z'\cdots)\\&\quad+\lambda\,T_{\Psi}(x'\cdots z'|yy'\cdots y^{(l)}|z'\cdots)\\&\quad-g(z')(-1)^l\,T\Psi(x'\cdots|yy'\cdots y^{(l)}|z'\cdots)\\&\quad+i\,\delta(y-z')(-1)^l\,T_{\Psi}(x'\cdots|y'\cdots y^{(l)}|z''\cdots)\\&+i\sum_{\eta\neq y}(\pm)\,\delta(\eta-z')\,T_{\Psi}(x'\cdots|yy'\cdots y^{(l)};\eta|z''\cdots)=0.\end{aligned}\right\}\quad\text{(Ap. III. 6)}$$

As the number of nucleon space time points has now increased by one, the third term has the required sign factor. The factor $(-1)^l$ appearing in the fourth term is in accordance with the convention as regards the value of the parity factor ($\pm$). Hence, the two last terms in (Ap. III. 6) combine to give a term of the form of the last term in (Ap. III. 5) and we see that (Ap. III. 6) is again of the form (Ap. III. 5).

The above equations have been derived by FREESE [4] for the source-free case by means of other methods.

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